

Moment-Fourier approach to parallel closures and transport for a toroidally confined multi-ion plasma

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We develop a multi-ion closure framework for tokamak plasmas by solving the drift-kinetic equation using the moment-Fourier method. The model extends our previous single-ion formulation to plasmas containing multiple ion species and consistently incorporates inter-species collisional effects in the moment system. As a representative case, we consider a deuterium-carbon plasma, assuming a common zeroth-order ion temperature on each flux surface while allowing the electron temperature to differ. The first-order fluid moments, namely density, temperature, and parallel flow velocity, are computed for electrons, deuterium ions, and carbon ions. The framework enables a direct comparison of the first-order parallel flow velocities of deuterium and carbon ions, thereby providing a quantitative basis for interpreting charge-exchange spectroscopy measurements and for systematic multi-ion neoclassical transport studies.

I. INTRODUCTION

Neoclassical transport theory provides a fundamental framework for understanding particle, heat, and momentum transport in magnetically confined plasmas due to Coulomb collisions in an inhomogeneous magnetic field [1–7]. The drift kinetic equation serves as the foundation for neoclassical transport calculations, and numerous numerical methods have been developed to solve it [8–11]. An analytic approach using the moment-Fourier method has been developed for ion-only plasmas [12, 13] and recently extended to electron-ion plasmas [14], enabling systematic derivation of both transport and closure relations with accurate Coulomb collision effects.

Multi-ion plasmas are ubiquitous in tokamak experiments, where impurity ions such as carbon play an important role in plasma transport and rotation. In particular, charge-exchange recombination spectroscopy (CXRS) provides direct measurements of impurity ion flow velocities [15], offering an experimental window into neoclassical transport. However, a systematic analytic framework for multi-ion neoclassical transport and closures based on accurate analytic Coulomb collision operators [16, 17] has not been available.

In this work, we extend the moment-Fourier method to multi-ion plasmas. As a representative case, we consider a deuterium-carbon-electron (D-C-e) plasma, incorporating inter-species collisional effects consistently in the moment system. The first-order fluid moments — density, temperature, and parallel flow velocity — are computed for all three species. The difference in parallel flow velocities between deuterium and carbon ions is examined as a function of plasma parameters, providing a quantitative basis for interpreting CXRS measurements and for systematic multi-ion neoclassical transport studies.

II. FOURIER-MOMENT APPROACH TO DRIFT KINETIC EQUATION

In this section, we present a system of parallel moment equations for a multi-ion plasma that generalizes the single-ion sys-

tem given in sections 2 and 3 of Ref. [14]. The key distinction from the single-ion and electron-ion cases lies in the collision operator C_{a1} [Eq. (4)], which now includes inter-species contributions from all pairs of species in addition to self-species collisions. These inter-species collision terms couple the moment equations for different ion species, requiring all species to be treated simultaneously in a unified linear system.

Although the formalism can be developed for a general magnetic field, in the present work we consider an axisymmetric circular magnetic field $\mathbf{B} = I(\psi)\nabla\zeta + \nabla\zeta \times \nabla\psi$ with $B = B_0/(1 + \epsilon \cos\theta)$, where ψ is a poloidal flux coordinate, ζ is the toroidal angle, and θ is the poloidal angle. In the formalism, we assume the zeroth-order distribution function to be Maxwellian,

$$f_{a0} = \frac{n_{a0}}{(2\pi T_{a0}/m_a)^{3/2}} e^{-w_a/T_{a0}} = \frac{n_{a0}}{\pi^{3/2} v_{a0}^3} e^{-s_a^2}, \quad (1)$$

where n_{a0} and T_{a0} are flux functions representing the zeroth-order density and temperature, $v_{a0} = \sqrt{2T_{a0}/m_a}$, and $s_a = v/v_{a0}$. Then we solve the following first-order drift kinetic equation

$$s_{a\parallel} \frac{\partial}{\partial\theta} f_{a1} - \frac{B}{B^\theta v_{a0}} C_{a1} = s_{a\parallel} \frac{\partial}{\partial\theta} F_a + \frac{B s_{a\parallel} E_{1\parallel}}{B^\theta T_{a0}} f_{a0}, \quad (2)$$

where

$$F_a = -\frac{I v_{\parallel}}{\Omega_a} \left[\frac{d \ln p_{a0}}{d\psi} + \left(s_a^2 - \frac{5}{2} \right) \frac{d \ln T_{a0}}{d\psi} + \frac{q_a}{T_{a0}} \frac{d\phi_0}{d\psi} \right] f_{a0}, \quad (3)$$

and

$$C_{a1} = \sum_b [C(f_{a1}, f_{b0}) + C(f_{a0}, f_{b1})] \quad (4)$$

with C denoting the Landau collision operator. Here $\phi_0(\psi)$ is the zeroth-order electric potential, and $E_{1\parallel}$ is the parallel component of the first-order electric field, which provides an independent drive for parallel flows in addition to the neoclassical drives contained in F_a [Eq. (3)].

In the moment-Fourier formulation, the first-order distribution function f_{a1} is expanded as

$$f_{a1} = f_{a0} \sum_{l=0}^{L-1} \sum_{q=0}^{K-1} \sum_{m=0}^{F-1} M_{a(m)}^{lq} \hat{P}_a^{lq} \varphi_{(m)}, \quad (5)$$

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where $\hat{P}_a^{lq} = s_a^l P_l(v_{\parallel}/v) L_q^{(l+1/2)}(s_a^2)/\sqrt{\sigma_{lq}}$, $s_a = v/v_{a0}$, σ_{lq} is the normalization constant for a Legendre polynomial P_l and an associated Laguerre polynomial $L_q^{(l+1/2)}$; and $\varphi_{(0)} = 1$, $\varphi_{(1)} = \varphi_{(1-)} = \sin \theta$, $\varphi_{(2)} = \varphi_{(1+)} = \cos \theta$, $\varphi_{(3)} = \varphi_{(2-)} = \sin 2\theta$, $\varphi_{(4)} = \varphi_{(2+)} = \cos 2\theta$, $\dots$. The expansion coefficients $M_{a(m)}^{lq}$ are the m -th Fourier component of the generalized velocity-space moment of f_{a1} . The lowest-order moments are identified with the first-order fluid variables: $M_a^{00} = n_{a1}/n_{a0}$, $M_a^{01} = -\sqrt{3/2} T_{a1}/T_{a0}$, $M_a^{10} = \sqrt{2} u_a/v_{a0}$, $M_a^{11} = -\sqrt{4/5} h_{a\parallel}/v_{a0} p_{a0}$, and $M_a^{20} = \sqrt{3/4} \pi_{a\parallel}/p_{a0}$, where n_{a1} , T_{a1} , u_a , $h_{a\parallel}$, and $\pi_{a\parallel}$ are the first-order density, temperature, parallel flow velocity, parallel heat flux density, and parallel viscosity, respectively.

Taking moments of Eq. (2) with $\hat{P}_a^{lp}$ yields a set of coupled ordinary differential equations (ODEs) in the poloidal angle θ . All θ -dependent quantities — the moments and the magnetic-field functions — are further expanded in Fourier series in θ , converting the ODE system to a block-structured set of linear algebraic equations for the Fourier coefficients $\{M_{a(m)}^{lq}\}$.

For a multi-ion plasma, the first-order collision term C_{a1} [Eq. (4)] couples the moment equations of species a to those of every other species b through inter-species Coulomb collisions. The resulting system for species a takes the form

$$D_a \mathbb{M}_a - \sum (A_{ab} \mathbb{M}_a + B_{ab} \mathbb{M}_b) = \mathbb{G}_a, \quad (6)$$

where D_a , A_{ab} , and B_{ab} are the LKF dimensional moment-Fourier matrices of $s_{a\parallel} \partial/\partial \theta$, $(B/B^\theta v_{a0}) C(f_{a1}, f_{b0})$, and $(B/B^\theta v_{a0}) C(f_{a0}, f_{b1})$, respectively; and $\mathbb{M}_a$ is the column vector of $\{M_{a(m)}^{lq}\}$; and $\mathbb{G}_a$ is the moment-Fourier vector of the right-hand side of Eq. (2) (F_a and $E_{1\parallel}$ terms). The full mass-ratio dependence is retained in all collision matrices without the small-mass ratio approximation. Although the approximation provides significantly reduced electron-ion and ion-electron collision matrices, it cannot be applied to ion-ion collisions consistently in a multi-ion plasma. Therefore it is convenient to adopt accurate collision operators for electrons and all ion species uniformly. Systems of the form of Eq. (6) for all species are assembled into a single linear system and solved simultaneously. The resulting first-order density, temperature, and parallel flow velocity for each species are expressed as linear combinations of the radial pressure and temperature gradients, $dp_{a0}/d\psi$ and $dT_{a0}/d\psi$, and the first-order parallel electric field $E_{1\parallel}$.

The closure relations for the density, temperature, and flow velocity equations can be obtained by solving Eq. (6) with the rows for $\{M_a^{00}, M_a^{01}, M_a^{10}\}$ moments removed. Due to the removal of the rows, the corresponding columns appear as closure drives in addition to F_a drive terms. The additional closure terms are parallel gradients of T_{a1} and u_a . Since the $E_{1\parallel}$ term has appeared only in the M_a^{10} equation, it does not appear in the system of equations for closures. Solving the system of equations yields the necessary moments to obtain closures, $\{Q_a, h_{a\parallel}, R_{a\parallel}, \pi_{a\parallel}\}$, where Q_a and $R_{a\parallel}$ are collisional heating and friction, respectively. The closure relations are expressed in terms of fluid variables in Fourier representation. Once the closure relations have been obtained, they can be combined with the linearized fluid equations to obtain n_{a1} , T_{a1} , and u_a . It is

obvious and we have verified that the fluid quantities obtained from the full transport system [Eq. (6)] agree with those obtained by combining the closure relations with the linearized fluid equations.

Letting $\mathbb{F}_a$ denote the Fourier column vector of n_{a1}/n_{a0} , T_{a1}/T_{a0} , and u_a/v_{a0} , the solution can be written as

$$\mathbb{F}_a = \sum_b \mathbb{F}_{ab}^p \hat{p}_{b0,\psi} + \sum_b \mathbb{F}_{ab}^T \hat{T}_{b0,\psi} + \sum_n \mathbb{F}_a^{E(m)} \hat{E}_{(m)}, \quad (7)$$

where

$$\hat{p}_{a0,\psi} = \frac{I}{q_a v_{a0} B_0} \left(\frac{1}{n_{a0}} \frac{dp_{a0}}{d\psi} + q_a \frac{d\phi_0}{d\psi} \right),$$

$$\hat{T}_{a0,\psi} = \frac{I}{q_a v_{a0} B_0} \frac{dT_{a0}}{d\psi},$$

and $\hat{E}_{(m)}$ is the m -th Fourier component of $\hat{E} = e E_{1\parallel} B/B^\theta T_{e0}$. The coefficient column vectors $\mathbb{F}_{ab}^p$, $\mathbb{F}_{ab}^T$, and $\mathbb{F}_a^{E(m)}$ encode the full neoclassical transport response of species a to the drives of species b , and the electric field. Then the first-order fluid quantities can be expressed in Fourier series $\sum_n \mathbb{F}_{a(n)} \varphi_{(n)}$ for each drive term $\hat{p}_{b0,\psi}$, $\hat{T}_{b0,\psi}$, and $\hat{E}_{(m)}$, where $\mathbb{F}_{a(n)}$ is the n -th Fourier coefficient.

III. RESULTS

We apply the framework to a D-C-e plasma with inverse aspect ratio $\epsilon = 0.1$. The electron and common ion temperatures are $T_e = 3 \text{ keV}$ and $T_D = T_C = 1.4 T_e$. Densities are $n_e = 2 \times 10^{19} \text{ m}^{-3}$, $n_D = 0.64 n_e$, and $n_C = 0.06 n_e$, which satisfy quasi-neutrality $n_D + Z_C n_C = n_e$ with $Z_C = 6$. The Knudsen number for the deuterium ions is 10^3 . The linear system (6) is constructed and solved for $L = 40$, $K = 80$, and $F = 9$. The resulting profiles of n_{a1} , T_{a1} , and u_a shown in Figs. 1–4 are reconstructed from the Fourier series $\sum_n \mathbb{F}_{a(n)} \varphi_{(n)}$, retaining $F = 9$ harmonics.

Figures 1 and 2 show the first-order density and temperature perturbations for D, C, and electrons driven separately by each neoclassical drive. The first-order density and temperature perturbations depend on the radial gradients of the zeroth-order densities and temperatures of all species and on the Fourier components of the parallel electric field. The contribution of D and C ion drives to the first-order electron density and temperature is negligible compared to the electron contribution. Figure 3 shows the parallel flow velocities driven by the pressure and temperature gradients. The D and C ion flows respond differently to each neoclassical drive, and the resulting flow velocity difference suggests that competing neoclassical drives may provide an explanation for the intrinsic rotation reversal. Figure 4 shows the parallel flow velocities driven by the parallel electric field $E_{1\parallel}$. For a circular magnetic field, the parallel flow velocity is driven only by the zeroth Fourier component $\hat{E}_{(0)}$, not by $\hat{E}_{(m)}$ for $m > 0$. The D and C velocities differ noticeably from each other due to the mass dependence in the inter-species collision matrices. Since C ion flows are directly measurable by CXRS, these results provide a basis for inferring D ion flows and for comparisons with experimental data.

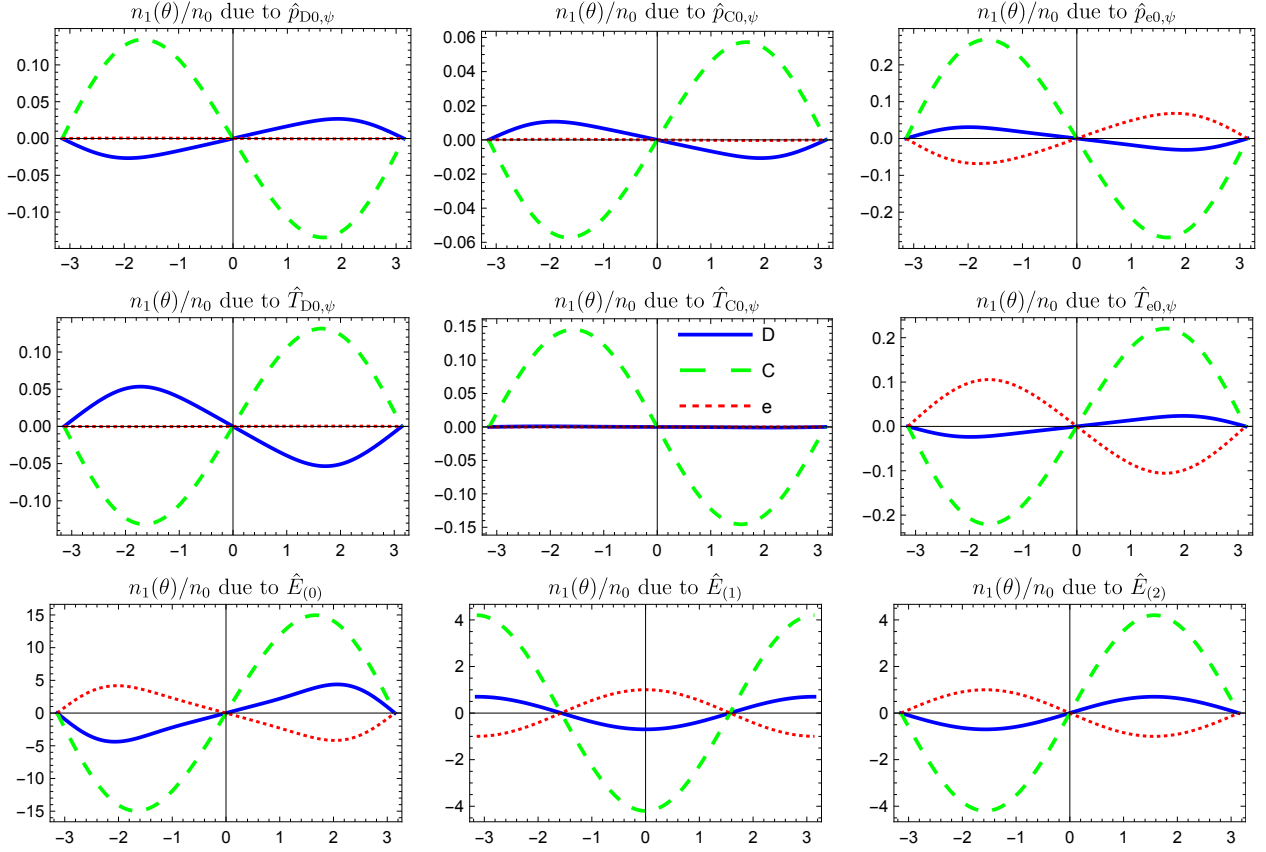


Figure 1. First-order density perturbation n_1/n_0 of D (solid, blue), C (dashed, green), and electrons (dotted, red) driven by the pressure gradient $\hat{p}_{b0,\psi}$, the temperature gradient $\hat{T}_{b0,\psi}$, ($b = D, C, e$), and the electric field $\hat{E}_{(m)}$ ($m = 0, 1, 2$) as functions of the poloidal angle θ .

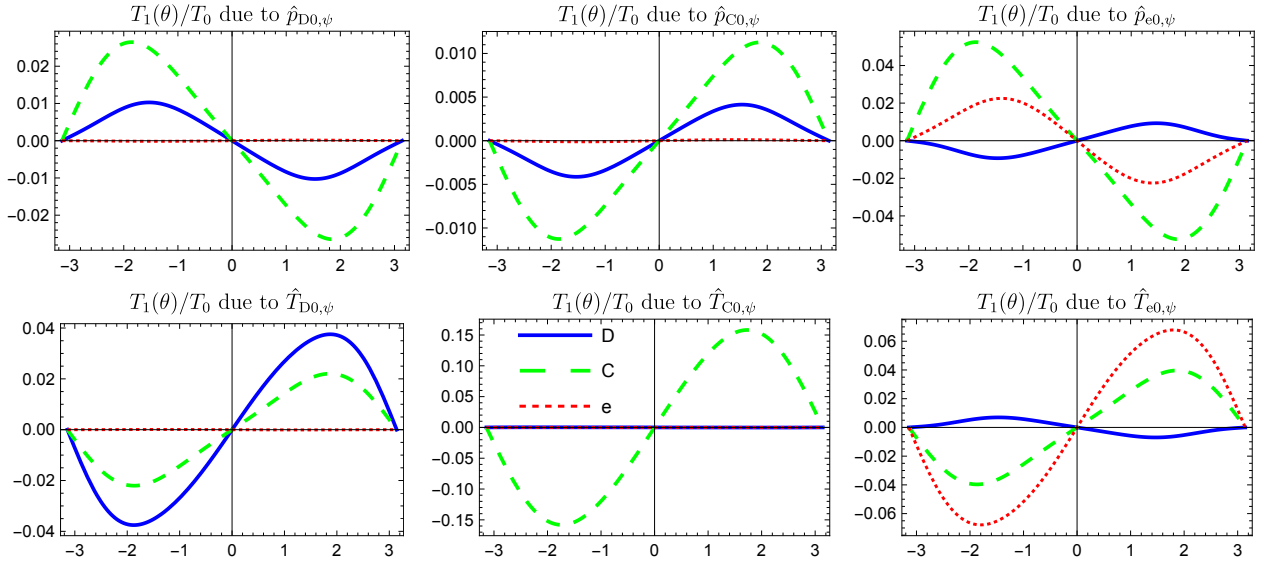


Figure 2. First-order temperature perturbation T_1/T_0 of D (solid, blue), C (dashed, green), and electrons (dotted, red) driven by the pressure gradient $\hat{p}_{b0,\psi}$ and the temperature gradient $\hat{T}_{b0,\psi}$.

IV. SUMMARY

We have extended the moment-Fourier method [13, 14] to multi-ion plasmas and applied it to a D-C-e plasma. The framework incorporates inter-species Coulomb collision effects con-

sistently in the moment-Fourier system without the small mass-ratio approximation, treating electrons and all ion species on the same footing in a single linear system. As an example study, we apply the method to a D-C-e plasma and compute the first-order density, temperature, and parallel flow velocity for

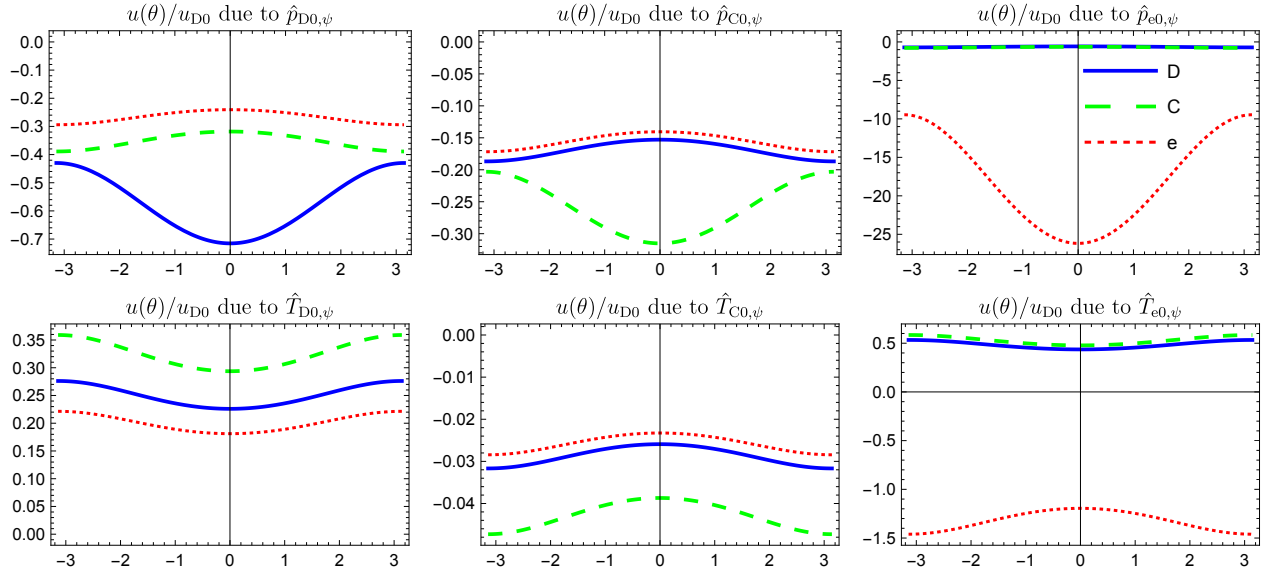


Figure 3. Parallel flow velocity u/u_{D0} of D (solid, blue), C (dashed, green), and electrons (dotted, red) driven by the pressure gradient $\hat{p}_{b0,\psi}$ and the temperature gradient $\hat{T}_{b0,\psi}$.

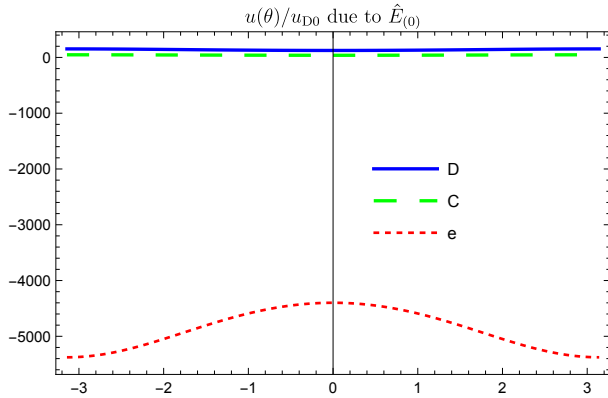


Figure 4. Parallel flow velocity u/u_{D0} of D (solid, blue), C (dashed, green), and electrons (dotted, red), driven by $E_{1||}$.

all three species, driven separately by the radial pressure and temperature gradients and the parallel electric field.

The D and C ion flows respond differently to each neoclassical drive, suggesting that competing drives may underlie the intrinsic rotation reversal. Understanding the flow difference between D and C ions also provides a basis for inferring D ion flows from CXRS measurements of C ions. Future work will address bootstrap current studies, intrinsic rotation analyses, and comparisons with experimental CXRS data.

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- [1] A. A. Galeev and R. Z. Sagdeev, *Sov. Phys. JETP* **26**, 233 (1968).
 - [2] P. H. Rutherford, *Phys. Fluids* **13**, 482 (1970), ISSN 0031-9171.
 - [3] F. L. Hinton and R. D. Hazeltine, *Rev. Mod. Phys.* **48**, 239 (1976).
 - [4] S. Hirshman and D. Sigmar, *Nucl. Fusion* **21**, 1079 (1981).
 - [5] C. S. Chang and F. L. Hinton, *Phys. Fluids* **25**, 1493 (1982).
 - [6] P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge, 2002).
 - [7] R. D. Hazeltine and J. D. Meiss, *Plasma Confinement* (Dover Pub., Inc., New York, 2003).
 - [8] E. A. Belli and J. Candy, *Plasma Phys. Control. Fusion* **50**, 095010 (2008).
 - [9] E. A. Belli and J. Candy, *Plasma Phys. Control. Fusion* **51**, 075018 (2009).
 - [10] E. A. Belli and J. Candy, *Plasma Phys. Control. Fusion* **54**, 015015 (2011).
 - [11] M. Landreman and D. R. Ernst, *Plasma Phys. Control. Fusion* **54**, 115006 (2012).
 - [12] S. K. Wong and V. S. Chan, *Plasma Phys. Control. Fusion* **53**, 095005 (2011).
 - [13] J.-Y. Ji, E. D. Held, J. A. Spencer, and Y.-S. Na, *Plasma Phys. Control. Fusion* **65**, 035018 (2023).
 - [14] J.-Y. Ji, M. U. Lee, Y.-S. Na, and E. D. Held, *Plasma Phys. Control. Fusion* **67**, 045034 (2025).
 - [15] C. Chrystal, K. H. Burrell, B. A. Grierson, R. J. Groebner, and D. H. Kaplan, *Rev. Sci. Instrum.* **83**, 10D501 (2012).
 - [16] J.-Y. Ji and E. D. Held, *Phys. Plasmas* **13**, 102103 (2006).
 - [17] J.-Y. Ji, J. A. Spencer, and E. D. Held, *Plasma Res. Express* **2**, 015013 (2020).