Linear Perturbative Ideal MHD Formulation and MHD Continuum Including Toroidal Rotation[1]

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Toroidal rotation effects in ideal magnetohydrodynamics (MHD) are treated in a manner fully consistent with their implementation in the existing version of the ideal MHD code-eigenmode NOVA. The system of equations originally derived and implemented in NOVA [1] is generalized here to include sheared toroidal plasma flow while preserving the original structure of the formulation and incorporating rotation-modified terms. Accurate treatment of radial and poloidal derivative terms entering through the rotating plasma equilibrium are essential for reliable simulation and interpretation of Alfvén eigenmode (AE) stability in fusion plasmas with strong neutral beam injection (NBI).

1. FORMULATION OF IDEAL MHD WITH TOROIDAL ROTATION

The ideal MHD stationary equilibrium equations in the presence of toroidal plasma rotation can be written as follows:

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right]=-\nabla p+\mathbf{J}\times\mathbf{B} \tag{1}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})=0 \tag{2}$$

$$\frac{\partial p}{\partial t}+\mathbf{V}\cdot\nabla p+\gamma_s p\nabla\cdot\mathbf{V}=0 \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{V}\times\mathbf{B}), \tag{4}$$

where $\mathbf{J}=\nabla\times\mathbf{B}$, $\nabla\cdot\mathbf{B}=0$, $\mathbf{J}$, $\mathbf{B}$ and p are the equilibrium current, magnetic field and plasma pressure. Pla sma equilibrium was considered as one member of the hierarchy of equilibria recently [2].

In the straight field line flux coordinate system (ψ,θ,ξ), the axisymmetric toroidal equilibrium magnetic field can be expressed as

$$\mathbf{B}=\nabla\zeta\times\nabla\psi+q(\psi)\nabla\psi\times\nabla\theta \tag{5}$$

where $2\pi\psi$ is the poloidal flux within a magnetic surface, θ is the generalized poloidal angle with a period of 2π, ζ is the generalized toroidal angle with a period of 2π, and q is the safety factor. The axisymmetric toroidal equilibrium magnetic field can also be written as $\mathbf{B}=\nabla\varphi\times\nabla\psi+g(\psi)\nabla\varphi$ where $g(\psi)$ is the toroidal field function. Therefore, $\mathbf{B}\cdot\nabla=\mathscr{J}^{-1}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\zeta}\right)=\mathscr{J}^{-1}\frac{\partial}{\partial\theta}+\frac{g}{R^2}\frac{\partial}{\partial\varphi}$, where $\mathscr{J}=[\nabla\psi\times\nabla\theta\cdot\nabla\zeta]^{-1}$ is the Jacobian.

In what follows, the definition of the perturbed plasma dislacement, magnetic field, and plasma pressure are $\boldsymbol{\xi}$, $\mathbf{b}$ and p_1, respectively. $\mathbf{V}$ is the purely toroidal flows of stability equilibria, i.e., $\mathbf{V}=R^2\Omega\nabla\varphi$, where $\Omega(\psi)$ is toroidal rotation fequency, see also Refs.[3–5].

The time dependence of perturbations is presented in the forms $\sim\exp(-i\omega t)$. Considering the toroidal symmetry in single Fourier modes, the displacement perturbation can be expressed by $\boldsymbol{\xi}\sim\exp(in\varphi)$, where n is the toroidal mode number. With the above ansatz, the linearized ideal MHD equations become

$$p_1+\boldsymbol{\xi}\cdot\nabla p+\frac{i}{\omega}\mathbf{V}\cdot\nabla p_1+\gamma_s p\nabla\cdot\boldsymbol{\xi}=0, \tag{6}$$

$$\rho\omega^2\boldsymbol{\xi}=\nabla p_1+\mathbf{b}\times[\nabla\times\mathbf{B}]+\mathbf{B}\times[\nabla\times\mathbf{b}]+\rho\left(\nabla(\mathbf{V}\cdot\boldsymbol{\xi})-\mathbf{V}\times[\nabla\times\boldsymbol{\xi}]-\boldsymbol{\xi}\times[\nabla\times\mathbf{V}]\right), \tag{7}$$

$$\mathbf{b}=\nabla\times(\boldsymbol{\xi}\times\mathbf{B})+\frac{i}{\omega}\nabla\times\left(R^2\Omega[\nabla\varphi\times\mathbf{b}]\right). \tag{8}$$

The plasma displacement and the magnetic field perturbation are expanded into three components:

$$\boldsymbol{\xi}=\frac{\xi_\psi}{|\nabla\psi|^2}\nabla\psi+\frac{\xi_s}{B^2}\mathbf{B}\times\nabla\psi+\frac{\xi_b}{B^2}\mathbf{B}, \tag{9}$$

$$\mathbf{b}=\frac{Q_\psi}{|\nabla\psi|^2}\nabla\psi+\frac{Q_s}{|\nabla\psi|^2}[\mathbf{B}\times\nabla\psi]+\frac{Q_b}{B^2}\mathbf{B}, \tag{10}$$

where $\xi_\psi=\boldsymbol{\xi}\cdot\nabla\psi$, $\xi_s=\boldsymbol{\xi}\cdot(\mathbf{B}\times\nabla\psi)/|\nabla\psi|^2$, $\xi_b=\boldsymbol{\xi}\cdot\mathbf{B}$, $Q_\psi=\mathbf{b}\cdot\nabla\psi$, $Q_s=\mathbf{b}\cdot(\mathbf{B}\times\nabla\psi)/B^2$ and $Q_b=\mathbf{b}\cdot\mathbf{B}$. Here, the subscript "$\psi$" denotes the perturbation component perpendicular to the magnetic flux surfaces, while the

[1] This work supported in part by DoE contract No. DE-AC02-76CH03073 and in part by National Natural Science Foundation of China (No. 12405260) and project ZR2024QA207 supported by Shandong Provincial Natural Science Foundation.

subscripts "b" and "s" correspond to the components on the magnetic surfaces that are parallel and perpendicular to the equilibrium magnetic field, respectively.

From Eq. 6, we find

$$p_1 = \frac{\tilde{\omega} - n\Omega}{\tilde{\omega}} \left(-p'\xi_\psi - \gamma_s p \nabla \cdot \boldsymbol{\xi}\right), \tag{11}$$

where $\nabla p = p' \nabla \psi$, γ_s is the specific heat ratio which is 5/3 in ideal MHD and has a more complicated form in the gyrokinetic approximation [6] and $\tilde{\omega} = \omega + n\Omega$. Then by multiplying $\nabla\psi$, $[\mathbf{B} \times \nabla\psi]/B^2$, and $\mathbf{B}$ by Eq.7, the following equations can be derived:

$$\rho\omega^2 \xi_\psi = \nabla\psi \cdot \nabla P_1 - |\nabla\psi|^2 \mathbf{B} \cdot \nabla \left(\frac{Q_\psi}{|\nabla\psi|^2}\right) - \left(|\nabla\psi|^2 S - \mathbf{B} \cdot \mathbf{J}\right) Q_s - 2K_\psi Q_b \tag{12}$$

$$- i\omega\rho \left[\Omega \frac{\partial \xi_\psi}{\partial \varphi} + \Omega \frac{|\nabla\psi|^2 \left(\nabla\psi \cdot \nabla R^2\right)}{B^2 R^2} \xi_s - \Omega \frac{g\left(\nabla\psi \cdot \nabla R^2\right)}{B^2 R^2} \xi_b\right],$$

$$\rho\omega^2 \frac{|\nabla\psi|^2}{B^2} \xi_s = \frac{\mathbf{B} \times \nabla\psi}{B^2} \cdot \nabla P_1 - \frac{(\mathbf{B} \cdot \mathbf{J})}{B^2} Q_\psi - \mathbf{B} \cdot \nabla Q_\psi - 2K_s Q_b \tag{13}$$

$$- i\omega\rho \left[\Omega \frac{|\nabla\psi|^2}{B^2} \frac{\partial \xi_s}{\partial \varphi} - \frac{\left(\nabla\psi \cdot \nabla R^2\right)}{B^2 R^2} \xi_\psi - \Omega \frac{\left(\mathbf{B} \cdot \nabla R^2\right)}{B^2} \xi_b\right],$$

$$\rho\omega^2 \xi_b = \mathbf{B} \cdot \nabla P_1 - \mathbf{B} \cdot \nabla Q_b + p' Q_\psi - i\omega\rho \left[\Omega \frac{\partial \xi_b}{\partial \varphi} - \frac{g\left(\nabla\psi \cdot \nabla R^2 \Omega\right)}{R^2 |\nabla\psi|^2} \xi_\psi - \Omega \left(\mathbf{B} \cdot \nabla R^2\right) \xi_b\right], \tag{14}$$

where $P_1 = p_1 + \mathbf{b} \cdot \mathbf{B}$ is the total perturbed pressure. $K_\psi = \mathbf{K} \cdot \nabla\psi$, $K_s = \mathbf{K} \cdot \left[(\mathbf{B} \times \nabla\psi)/B^2\right]$, and $\mathbf{K} = (\mathbf{B}/B) \cdot \nabla(\mathbf{B}/B)$ is the curvature vector. $S = \left[(\mathbf{B} \times \nabla\psi)/|\nabla\psi|^2\right] \cdot \nabla \times \left[(\mathbf{B} \times \nabla\psi)/|\nabla\psi|^2\right]$ is the negative local magnetic shear.

Similarly, using $\nabla\psi$, $\left(\mathbf{B} \times \nabla\psi/|\nabla\psi|^2\right)$, and $\mathbf{B}/B^2$ to multiply Eq.10, we can derive that

$$Q_\psi = \frac{\tilde{\omega} - n\Omega}{\tilde{\omega}} \mathbf{B} \cdot \nabla \xi_\psi, \tag{15}$$

$$Q_s = \frac{(\tilde{\omega} - n\Omega)|\nabla\psi|^2}{\tilde{\omega} B^2} \left[\mathbf{B} \cdot \nabla \xi_s - S\xi_\psi - i\frac{\Omega'}{\tilde{\omega}} \mathbf{B} \cdot \nabla \xi_\psi\right], \tag{16}$$

$$Q_b = \frac{(\tilde{\omega} - n\Omega)}{\tilde{\omega}} \left[B^2 \mathbf{B} \cdot \nabla \frac{\xi_b}{B^2} - B^2 \nabla \cdot \boldsymbol{\xi} - 2K_s B^2 \xi_s - 2K_\psi \frac{B^2}{|\nabla\psi|^2} \xi_\psi + p'\xi_\psi + ig\frac{\Omega'}{\tilde{\omega}} \mathbf{B} \cdot \nabla \xi_\psi\right], \tag{17}$$

where $\tilde{\omega} = \omega + i\mathbf{V} \cdot \nabla = \omega + n\Omega$. Also, $\nabla \cdot \boldsymbol{\xi}$ can be expanded according to the following equation

$$\nabla \cdot \boldsymbol{\xi} = \frac{\nabla\psi \cdot \nabla \xi_\psi}{|\nabla\psi|^2} + \left[\nabla \cdot \left(\frac{\nabla\psi}{|\nabla\psi|^2}\right)\right] \xi_\psi + \frac{\mathbf{B} \times \nabla\psi \cdot \nabla \xi_s}{B^2} - 2K_s \xi_s + \mathbf{B} \cdot \nabla \left(\frac{\xi_b}{B^2}\right) \quad . \tag{18}$$

By solving the coupled system of equations given in Eqs.12–18, the final set of equations suitable for implementation in NOVA framework, can be written in the original form of [1] the following way:

$$\nabla\psi \cdot \nabla \begin{pmatrix} P_1 \\ \xi_\psi \end{pmatrix} = \mathbf{C}^s \begin{pmatrix} P_1 \\ \xi_\psi \end{pmatrix} + \mathbf{D}^s \begin{pmatrix} \xi_s \\ \nabla \cdot \xi \end{pmatrix}, \tag{19}$$

and

$$\mathbf{E}^s \begin{pmatrix} \xi_s \\ \nabla \cdot \xi \end{pmatrix} = \mathbf{F}^s \begin{pmatrix} P_1 \\ \xi_\psi \end{pmatrix}. \tag{20}$$

The superscript s is introduced for the matrix components where the sheared toroidal rotation is included. Here

$$\begin{aligned} C^s_{11} = 2K_\psi \quad C^s_{12} &= \frac{(\tilde{\omega}-n\Omega)}{\tilde{\omega}} \left[G^s - \frac{\rho\Omega g^2\left(\nabla\psi\cdot\nabla R^2\right)\left(\nabla\psi\cdot\nabla R^2\Omega\right)}{B^2 R^4 |\nabla\psi|^2}\right] \\ C^s_{21} = 0 \quad C^s_{22} &= |\nabla\psi|^2 \left[-\left(\nabla \cdot \frac{\nabla\psi}{|\nabla\psi|^2}\right) + \frac{ig}{\tilde{\omega}} \mathbf{B} \cdot \nabla \left(\frac{\left(\nabla\psi\cdot\nabla R^2\Omega\right)}{B^2 R^2 |\nabla\psi|^2}\right)\right] \end{aligned} \tag{21}$$

$$G^s = \rho\tilde{\omega}^2 + 2K_\psi p' + |\nabla\psi|^2 \mathbf{B} \cdot \nabla \left(\frac{\mathbf{B}\cdot\nabla}{|\nabla\psi|^2}\right) + \left(\mathbf{B} \cdot \mathbf{J} - |\nabla\psi|^2 S\right) \frac{|\nabla\psi|^2}{B^2} \left(S + \frac{i\Omega'}{\tilde{\omega}} \mathbf{B} \cdot \nabla\right) \tag{22}$$

$$\begin{aligned} D^s_{11} &= \frac{(\tilde{\omega}-n\Omega)}{\tilde{\omega}} \left[\left(|\nabla\psi|^2 S - \mathbf{B} \cdot \mathbf{J}\right) \frac{|\nabla\psi|^2}{B^2} \mathbf{B} \cdot \nabla + \frac{i\rho\tilde{\omega}\Omega\left(\nabla\psi\cdot\nabla R^2\right)|\nabla\psi|^2}{B^2 R^2} - \frac{\rho\Omega^2 g\left(\nabla\psi\cdot\nabla R^2\right)\left(\mathbf{B}\cdot\nabla R^2\right)}{B^2 R^2}\right] \\ D^s_{12} &= \frac{(\tilde{\omega}-n\Omega)}{\tilde{\omega}} \left[2K_\psi \gamma_s p + \frac{i\gamma_s p\Omega g\left(\nabla\psi\cdot\nabla R^2\right)}{\tilde{\omega} B^2 R^2} \mathbf{B} \cdot \nabla\right] \\ D^s_{21} &= |\nabla\psi|^2 \left[2K_s - \frac{\mathbf{B}\times\nabla\psi}{B^2} \cdot \nabla + \frac{i\Omega}{\tilde{\omega}} \mathbf{B} \cdot \nabla \left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)\right] \\ D^s_{22} &= |\nabla\psi|^2 \left[1 + \frac{\gamma_s p}{\tilde{\omega}^2} \mathbf{B} \cdot \nabla \left(\frac{\mathbf{B}\cdot\nabla}{B^2 \rho}\right)\right] \end{aligned} \tag{23}$$

$$
\begin{aligned}
E^s_{11} &= \rho\tilde{\omega}^2\frac{|\nabla\psi|^2}{B^2}+\mathbf{B}\cdot\nabla\left[\frac{|\nabla\psi|^2}{B^2}(\mathbf{B}\cdot\nabla)\right]-\frac{\rho\Omega^2}{B^2}\left(\mathbf{B}\cdot\nabla R^2\right)^2\\
E^s_{12} &= 2\gamma_s pK_s+\frac{i\gamma_s p\Omega}{\tilde{\omega}B^2}\mathbf{B}\cdot\nabla R^2\mathbf{B}\cdot\nabla\\
E^s_{21} &= 2K_s+\frac{i\Omega}{\tilde{\omega}}\mathbf{B}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right);\quad E^s_{22}=\frac{\gamma_s p}{B^2}+\frac{\gamma_s p}{\tilde{\omega}^2\rho}\mathbf{B}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla}{B^2}\right)+1
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
F^s_{11} &= \frac{\tilde{\omega}}{\tilde{\omega}-n\Omega}\left(-2K_s+\frac{\mathbf{B}\times\nabla\psi}{B^2}\cdot\nabla\right)\\
F^s_{12} &= \mathbf{B}\cdot\nabla\left(\frac{|\nabla\psi|^2}{B^2}S\right)-\left(\frac{\mathbf{B}\cdot\mathbf{J}}{B^2}\right)\mathbf{B}\cdot\nabla-2K_s p'+i\frac{\Omega'}{\tilde{\omega}}\mathbf{B}\cdot\nabla\left(\frac{|\nabla\psi|^2}{B^2}\mathbf{B}\cdot\nabla\right)\\
&+ i\tilde{\omega}\rho\frac{(\nabla\psi\cdot\nabla R^2\Omega)}{B^2R^2}+\frac{\rho\Omega g(\mathbf{B}\cdot\nabla R^2)(\nabla\psi\cdot\nabla R^2\Omega)}{B^2R^2|\nabla\psi|^2}\\
F^s_{21} &= -\frac{\tilde{\omega}}{(\tilde{\omega}-n\Omega)}\frac{1}{B^2};\quad F^s_{22}= -\frac{2K_\psi}{|\nabla\psi|^2}+ig\frac{\Omega'}{\tilde{\omega}B^2}\mathbf{B}\cdot\nabla-\frac{ig}{\tilde{\omega}}\mathbf{B}\cdot\nabla\left(\frac{\nabla\psi\cdot\nabla R^2\Omega}{B^2R^2|\nabla\psi|^2}\right),
\end{aligned}
\tag{25}
$$

where the prefactor $(\tilde{\omega}-n\Omega)/\tilde{\omega}$ cancels out in several matrix elements such as E^s_{ij}. The above expressions reduce trivially to the original NOVA formulation in the limit of vanishing rotation, $\Omega\to 0$. We also note that, in the absence of rotation, similar representations of the perturbed variables are employed in other established linear ideal MHD codes such as PEST [7] and KINX [8].

2. IDEAL MHD CONTINUUM OVERVIEW

We present the following analysis using the standard ansatz exploiting the toroidal axisymmetry, $\xi(\theta,\zeta,t)=\xi_0(\theta,t)e^{-in\zeta}$, where $\zeta=\varphi-q\delta(\theta,\psi)$ and δ is periodic in θ [1]. Our analysis provides a self-consistent ideal-MHD description of the Alfvén continuum associated with the solutions at the resonant radial locations.

The continuum spectrum equation is expressed against NOVA chosen variables, ξ_s and $\nabla\cdot\boldsymbol{\xi}$ as follows from (20):

$$
\mathbf{E}^s\begin{pmatrix}\xi_s\\ \nabla\cdot\boldsymbol{\xi}\end{pmatrix}=0,
\tag{26}
$$

where the standard flux coordinates are adopted. Eq.(26) represents the locations of vanishing highest derivative coefficients, which occur for the second derivative in the case of MHD, Eq.(19). The Doppler-shifted contribution to Aa continuum enters primarily through a transformation, $\tilde{\omega}=\omega+n\Omega(\psi)$, i.e. the rotation frequency enters as a local value at each surface. To proceed further, we write the following relation: $\mathbf{B}\cdot\nabla\boldsymbol{\xi}=\mathcal{J}^{-1}\left(\frac{\partial}{\partial\theta}-inq\right)\xi_0e^{-in\zeta}=\mathcal{J}^{-1}e^{inq\theta}\frac{\partial}{\partial\theta}\left(\xi e^{-inq\theta}\right)$. We introduce new functions $Y_1(\theta)=\xi_s\exp[in(\zeta-q\theta)]$ and $Y_2(\theta)=\nabla\cdot\boldsymbol{\xi}\exp[in(\zeta-q\theta)]$. Then, using Eqs.26, and highlighting the terms associated with toroidal rotation in blue, we obtain in the vicinity of the rational surface:

$$
\left\{\begin{matrix}\rho\tilde{\omega}^2\frac{|\nabla\psi|^2}{B^2}+\frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}\frac{|\nabla\psi|^2}{B^2\mathcal{J}}\frac{\partial}{\partial\theta}-\rho\Omega^2\frac{\left(\mathbf{B}\cdot\nabla R^2\right)^2}{B^2} & \gamma_s p\left[2K_s+i\frac{\Omega}{\tilde{\omega}}\frac{(\mathbf{B}\cdot\nabla R^2)}{B^2\mathcal{J}}\frac{\partial}{\partial\theta}\right]\\ 2K_s+i\frac{\Omega}{\tilde{\omega}\mathcal{J}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right) & 1+\frac{\gamma_s p}{B^2}\left[1+\frac{B^2}{\tilde{\omega}^2\rho\mathcal{J}}\frac{\partial}{\partial\theta}\left(\frac{1}{B^2\mathcal{J}}\frac{\partial}{\partial\theta}\right)\right]\end{matrix}\right\}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}=0.
\tag{27}
$$

To elucidate the fundamental properties of the Alfvén–acoustic (Aa) continuum, we consider a simplified form of the continuum equation, (27) by assuming that only one m poloidal harmonic is present and by retaining the poloidal derivative terms acting on the perturbed variables $Y_{1,2}$. One obtains an approximate set of ideal MHD Aa continuum equations. This reduced description captures the local continuum dynamics and makes the origin of the continuum singularities and the Alfvén–acoustic coupling more transparent:

$$
\left\{\begin{matrix}\tilde{\omega}^2-k_\parallel^2v_A^2-\Omega^2 & v_a^2\left(2K_s-\frac{\Omega}{\tilde{\omega}}k_\parallel\right)\\ \tilde{\omega}^2\left(2K_s-\frac{\Omega}{\tilde{\omega}}k_\parallel\right) & \tilde{\omega}^2+\frac{\beta\gamma_s}{2}\left(\tilde{\omega}^2-v_A^2k_\parallel^2\right)\end{matrix}\right\}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}\simeq 0,
\tag{28}
$$

where $k_\parallel=(m-nq)/qR$ and $v_a^2=v_A^2\gamma_s\beta/2$. As can be seen, the Doppler-shifted frequency $\tilde{\omega}=\omega+n\Omega$ alone does not fully represent the effects of sheared toroidal rotation [9, 10]. The exact formulation contains additional rotation-dependent terms arising from the rotating equilibrium, which alter the continuum structure beyond the simple Doppler-shift approximation commonly employed in low-rotation theories.

We expect that the main Aa continuum properties described in Ref.[3] are captured by this schematic approach. This set of equations is useful to understand where each oscillation, Alfvénic or acoustic, is coming from and to avoid the potential confusion in the interpretations [11]. From Eq.(28) in the schematic manner we find that the Aa continuum system of equations satisfies

$$
\left(\tilde{\omega}^2-k_\parallel^2v_A^2-\Omega^2\right)\left[\tilde{\omega}^2+\frac{\beta\gamma_s}{2}\left(\tilde{\omega}^2-v_A^2k_\parallel^2\right)\right]-v_a^2\tilde{\omega}^2\left(2K_s-\frac{\Omega}{\tilde{\omega}}k_\parallel\right)^2\simeq 0.
\tag{29}
$$

If $\Omega\to 0$ and $\omega^2\ll v_A^2k_\parallel^2$ that equation reduces to acoustic branch, $\omega^2=(\beta\gamma_s/2)v_A^2k_\parallel^2$. On the other hand if $\Omega\to 0$ and $\omega^2\sim v_A^2k_\parallel^2$ we recover the shear Alfvén branch with $\omega^2=v_A^2k_\parallel^2$. The above considerations are accurate to $O(\gamma\beta,\Omega)$. The second term on the left-hand side represents the coupling between the acoustic and Alfvén branches and therefore determines the strength of the Alfvén–acoustic interaction. This coupling is responsible for the modification of the continuum spectrum and the appearance of Alfvén–acoustic gaps under appropriate plasma conditions.

Another point is that the first multiplier on the LHS, i.e. $\left(\tilde{\omega}^2 - k_{\parallel}^2 v_A^2 - \Omega^2\right)$, accounts for the continuum gap at zero frequency due to the toroidal rotation induced Doppler shift only, $\tilde{\omega} = \omega + n\Omega \simeq \pm\sqrt{k_{\parallel}^2 v_A^2 + \Omega^2} \simeq \Omega$.

3. LAGRANGIAN APPROACH TO ALFVÉN-ACOUSTIC CONTINUUM

Slow-sound approximation (SSA) [12] is obtained from Eq.(27) by neglecting the effect of the second derivative on the function Y_2. This approximation—also known as the Chu filtering scheme—represents the lowest-order consistent approximation that retains acoustic-wave effects.

The principal advantage of the Lagrangian approach to the continuum is that the resonance point with the continuum and corresponding singularity locations are determined self-consistently from the full set of equations rather than through asymptotic expansions. In contrast, approaches such as Ref.[3] are derived under the assumptions $\varepsilon = r/R_0 \ll 1$ and $\beta \ll 1$, which may limit their accuracy in finite-aspect-ratio, high-β, or strongly rotating plasmas. The present formulation avoids these restrictions and therefore provides a more general description of the Alfvén–acoustic continuum.

Instead of developing the full Lagrangian let us adopt the SSA now by modifying the system (27). This is done by neglecting the second derivative term in θ in the second equation:

$$\left\{ \begin{array}{cc} \rho\tilde{\omega}^2 \frac{|\nabla\psi|^2}{B^2} + \frac{1}{\mathscr{J}}\frac{\partial}{\partial\theta}\frac{|\nabla\psi|^2}{B^2\mathscr{J}}\frac{\partial}{\partial\theta} - \rho\Omega^2\frac{\left(\mathbf{B}\cdot\nabla R^2\right)^2}{B^2} & \gamma_s p\left[2K_s + i\frac{\Omega}{\tilde{\omega}}\frac{\left(\mathbf{B}\cdot\nabla R^2\right)}{B^2\mathscr{J}}\frac{\partial}{\partial\theta}\right] \\ 2K_s + i\frac{\Omega}{\tilde{\omega}\mathscr{J}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right) & 1 + \frac{\gamma_s p}{B^2} \end{array} \right\} \left(\begin{array}{c} Y_1 \\ Y_2 \end{array} \right) = 0. \tag{30}$$

Further, introducing a new function $Z_s \equiv \frac{\gamma_s p}{\mathscr{J}\tilde{\omega}^2\rho B^2}\frac{\partial Y_2}{\partial\theta}$, we find

$$\left\{ \begin{array}{ll} \mathscr{J}\left[\rho\tilde{\omega}^2\frac{|\nabla\psi|^2}{B^2} - \rho\Omega^2\frac{\left(\mathbf{B}\cdot\nabla R^2\right)^2}{B^2}\right]Y_1 + \frac{\partial}{\partial\theta}\frac{|\nabla\psi|^2}{B^2\mathscr{J}}\frac{\partial}{\partial\theta}Y_1 - \mathscr{J}\frac{4\gamma_s p K_s^2 B^2}{B^2+\gamma_s p}Y_1 - \frac{2\gamma_s p K_s B^2}{B^2+\gamma_s p}\left[i\frac{\Omega}{\tilde{\omega}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)Y_1\right] & \\ \quad + \frac{\gamma_s p}{\tilde{\omega}^2}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)\frac{\partial}{\partial\theta}\left[\frac{B^2}{B^2+\gamma_s p}\frac{\Omega^2}{\mathscr{J}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)\right]Y_1 - i\Omega\frac{\gamma_s p}{\tilde{\omega}B^2}\left(\mathbf{B}\cdot\nabla R^2\right)\frac{\partial}{\partial\theta}\left[\frac{B^2}{B^2+\gamma_s p}2K_s Y_1\right] & = 0 \\ \quad - \frac{\gamma_s p}{\mathscr{J}\tilde{\omega}^2\rho B^2}\frac{\partial}{\partial\theta}\left[\frac{B^2}{B^2+\gamma_s p}i\frac{\Omega}{\tilde{\omega}\mathscr{J}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)\right]Y_1 - \frac{\gamma_s p}{\mathscr{J}\tilde{\omega}^2\rho B^2}\frac{\partial}{\partial\theta}\left[\frac{B^2}{B^2+\gamma_s p}2K_s Y_1\right] & = Z_s \end{array} \right. \tag{31}$$

The resulting Lagrangian admits both the exact continuum solution and the SSA limit. The latter is

$$\mathscr{L}_s = \oint\left\{\mathscr{J}\tilde{\omega}^2\rho\left(\frac{|\nabla\psi|^2}{B^2} - \frac{\Omega^2}{\tilde{\omega}^2}\frac{\left(\mathbf{B}\cdot\nabla R^2\right)^2}{B^2}\right)|Y_1|^2 - \frac{|\nabla\psi|^2}{B^2\mathscr{J}}\left|\frac{\partial Y_1}{\partial\theta}\right|^2 - \mathscr{J}\frac{4\gamma_s p K_s^2 B^2}{B^2+\gamma_s p}|Y_1|^2\right. \tag{32}$$

$$-Y_1^*\frac{2\gamma_s p K_s B^2}{B^2+\gamma_s p}\left[i\frac{\Omega}{\tilde{\omega}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)Y_1\right] - Y_1^*\frac{i\Omega\gamma_s p}{\tilde{\omega}B^2}\left(\mathbf{B}\cdot\nabla R^2\right)\frac{\partial}{\partial\theta}\left[\frac{B^2}{B^2+\gamma_s p}2K_s Y_1\right] \tag{33}$$

$$\left.+Y_1^*\frac{\gamma_s p}{\tilde{\omega}^2}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)\frac{\partial}{\partial\theta}\left[\frac{B^2}{B^2+\gamma_s p}\frac{\Omega^2}{\mathscr{J}}\frac{\partial}{\partial\theta}\left(\frac{\mathbf{B}\cdot\nabla R^2}{B^2}\right)\right]Y_1\right\}d\theta. \tag{34}$$

The above Lagrangian is nonhermitian due to the presence of the toroidal flow. Nevertheless it can be solved by its minimization technique.

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