Model of low-frequency MHD waves at the plasma pedestal

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An intermediate plasma state, with better confinement than the L-mode but not yet a fully developed H-mode and free of type-I ELMs, has been routinely reported during the L-H transition at several fusion devices (JET, AUG, DIII-D, EAST, etc), providing an interesting option for ITER operation at marginal heating power [1]. Low-frequency (~ 1 kHz) axisymmetric ($n=0$, $m=\pm 1$) magnetic oscillations are often detected at the top of the pedestal by a variety of diagnostics for this specific confinement state, which is referred to as I-phase at AUG [2] and M-mode at JET [3]. The precise nature of these low-frequency oscillations (LFOs) is still elusive, regardless of several experimental scaling laws [2, 3, 4] and theoretical models [3, 4] recently proposed to predict their frequency values.

In this work, a local dispersion relation for LFOs, originally introduced by analogy with hydrodynamic internal waves propagating in stratified neutral fluids [3, 5], is rigorously derived within the framework of ideal MHD theory for a simplified slab geometry. It predicts Alfvén waves oscillating at frequencies much smaller than the typical value $\omega_A = k_\parallel v_A$ (i.e., of magnetic-shear waves, with v_A and $k_\parallel$ the Alfvén speed and the parallel wave vector, respectively) that depend strongly on the local characteristic length scales of the sharply inhomogeneous magnetic field, density, and temperature profiles at the plasma pedestal. The conditions for the propagation or damping of such waves are discussed, along with their relevance for the interpretation of the LFOs observed in fusion experiments.

Besides having a frequency lower than that expected for Alfvén waves ($\sim 10^2$ kHz) by two orders of magnitude, LFOs reported at JET are also distinctive due to a similar ratio between their characteristic length scales along the radial ($\lambda^{-1} \sim 10^{-2}$ m, as measured by correlation reflectometry) and poloidal ($k_{\mathrm{pol}}^{-1} \sim a/m \sim 1$ m, where a is the minor radius) directions [3]. Because the plasma beta $\beta = c_S^2/v_A^2 \sim 10^{-4}$ (here the ratio of the squared sound and Alfvén velocities) is very small at the pedestal, the space and time scales of the LFOs may be conveniently summarised by the orderings

$$\omega^2 \sim \omega_S^2 \sim \beta\,\omega_A^2 \quad \text{and} \quad k^2/\lambda^2 \sim \beta \ll 1. \tag{1}$$

Above, $\omega_A = k_\parallel v_A$ and $\omega_S = k_\parallel c_S$ are the Alfvén and sound frequencies, with $k_\parallel = \boldsymbol{k}\cdot\boldsymbol{B}/B \approx (m/q+n)/R_0$, $\boldsymbol{B}$ the magnetic field, q the safety factor, and $k^2 \approx (m/r)^2 + (n/R_0)^2$, where r

is the distance to the magnetic axis at the major radius R_0. LFOs are thus nearly sonic and sub-Alfvénic regarding their frequency, while keeping a very low ratio between radial and poloidal length scales. Also, their axisymmetry leads to small parallel wave numbers

$$k_{\parallel}^2/k^2 \sim a^2\big/\big(R_0^2 q^2\big) \ll 1. \tag{2}$$

A remarkable linear relationship between measured frequency and the poloidal Alfvén speed $v_{A,\mathrm{pol}} = (B_{\mathrm{pol}}/B)v_A$ was reported for LFOs in ICRF-heated deuterium plasmas at JET [3], hinting at their nature as shear Alfvén waves. Indeed, the condition $n = 0$ implies that $\omega \approx k_{\parallel} v_A \longrightarrow k_{\mathrm{pol}} v_{A,\mathrm{pol}}$, but predicted frequencies are then much larger than measurements. By analogy with internal waves in vertically stratified fluids with density $\rho(z)$ under a gravitational field $-g\,\mathbf{e}_z$, which have their Brunt-Väisälä frequency reduced by the ratio $k_{\mathrm{horiz}}/k_{\mathrm{vert}}$ if $k_{\mathrm{horiz}} \ll k_{\mathrm{vert}}$ [5], a modified shear-wave dispersion relation was conjectured as [3]

$$\omega^2 = \underbrace{-g\frac{\rho'}{\rho}}_{\text{Brunt-Väisälä frequency}} \frac{k_{\mathrm{horiz}}^2}{k_{\mathrm{vert}}^2} \longrightarrow \omega^2 = \underbrace{k_{\mathrm{pol}}^2 v_{\mathrm{A,pol}}^2}_{\text{shear-wave frequency}} \frac{k_{\mathrm{pol}}^2}{\lambda^2}. \tag{3}$$

This conjectured dispersion relation accounts for the necessary reduction of the predicted frequency by the ratio $k_{\mathrm{pol}}/\lambda \ll 1$ and is next derived in the framework of ideal MHD.

The linearised equations relating the pressure and field perturbations, p_1 and $\boldsymbol{B}_1$, with the plasma displacement ξ are

$$p_1 = -\xi\cdot\nabla p - \gamma p(\nabla\cdot\xi), \tag{4}$$

$$\boldsymbol{B}_1 = (\boldsymbol{B}\cdot\nabla)\xi - (\xi\cdot\nabla)\boldsymbol{B} - (\nabla\cdot\xi)\boldsymbol{B}, \tag{5}$$

$$\mu_0\rho\,\partial_{tt}^2\xi = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}_1 + (\boldsymbol{B}_1\cdot\nabla)\boldsymbol{B} - \nabla(\mu_0\tilde{p}), \tag{6}$$

where $\tilde{p} = p_1 + \mu_0^{-1}\boldsymbol{B}\cdot\boldsymbol{B}_1$ is the total pressure perturbation. Assuming a slab geometry to discard distracting geometric effects, along with inhomogeneous background density and magnetic field, respectively $\rho(x)$ and $\boldsymbol{B}(x) = B_y(x)\boldsymbol{e}_y + B_z(x)\boldsymbol{e}_z$, the seven linear equations above reduce to the system of differential equations

$$\tilde{p}' = S(x)\xi_x \quad \text{and} \quad S(x)\xi_x' = I(x)\tilde{p}, \tag{7}$$

coupling $\tilde{p}$ with the displacement component along the inhomogeneous direction, written as $\xi_x(x)e^{i(k_y y + k_z z - \omega t)}$, via the functions

$$S = \rho\big(\omega^2 - \omega_A^2\big) \quad \text{and} \quad I = k^2 + k_{\parallel}^2\big(\omega^4/\omega_A^2\big)\big/\big[\omega_S^2 - \big(1+\beta\big)\omega^2\big]. \tag{8}$$

Equations (7) are the slab-geometry limit of the known cylindrical system [6], yielding

$$\tilde{p}'' - \big(S'\big/S\big)\tilde{p}' - I\tilde{p} = 0 \quad \text{and} \quad \xi_x'' + \big(S'/S - I'/I\big)\xi_x' - I\xi = 0. \tag{9}$$

In order to relate the frequency with relevant length scales, let all variables be as

$$\tilde{p}(x) = \tilde{p}(0)e^{-\lambda x}, \quad \omega_A^2(x) = \omega_A^2(0)e^{x/L_A}, \quad \cdots, \tag{10}$$

with constant local length scales defined as

$$1/L_A = \omega_A^{2'}/\omega_A^2, \quad 1/L_{kB} = (\boldsymbol{k}\cdot\boldsymbol{B})^{2'}/(\boldsymbol{k}\cdot\boldsymbol{B})^2, \quad \cdots. \tag{11}$$

After replacing into the wave equation for $\tilde{p}$, one gets a series in the small parameter $\beta \ll 1$,

$$\underbrace{1 + \frac{1}{\lambda L_{kB}}}_{\sim\beta^0} + \underbrace{\beta\left(\frac{\tilde{\omega}^2}{\lambda L_A} - K^2 - K_\parallel^2 \frac{\beta\tilde{\omega}^4}{1-\tilde{\omega}^2}\right)}_{\sim\beta^1} + \cdots = 0, \tag{12}$$

where all constants $\tilde{\omega}^2 = \omega^2/\omega_S^2$, $K_\parallel^2 = k_\parallel^2/(\lambda^2\beta)$, $K^2 = k^2/(\lambda^2\beta) \sim 1$ are evaluated at $x = 0$. Now, terms $O(1)$ set the inverse length scale $\lambda = -1/L_{kB}$, whilst $O(\beta)$ ones yield a quadratic dispersion relation for $\tilde{\omega}^2$. Its two algebraic roots correspond to the frequency branches

$$\tilde{\omega}_\sigma^2 = 1 - \beta\lambda L_A K_\parallel^2 \big/ \big(1 - \lambda L_A K^2\big) + \cdots \quad \text{and} \tag{13}$$

$$\tilde{\omega}_\alpha^2 = \lambda L_A K^2 \Big[1 + \beta\lambda^2 L_A^2 K_\parallel^2 K^2 \big/ \big(1 - \lambda L_A K^2\big) + \cdots\Big], \tag{14}$$

the former being a very slightly modified sound wave and the latter bearing a shape quite similar to the one conjectured in (3), if denormalised first as $\omega_\alpha^2 = \lambda L_A \omega_A^2 \big(k^2/\lambda^2\big) + O(\beta)$ and then written, for the particular case $n = 0$, as $\omega_\alpha^2 = \lambda L_A k_{\text{pol}}^2 v_{A,\text{pol}}^2 \big(k_{\text{pol}}^2/\lambda^2\big) + O(\beta)$.

The layout of the relevant length scales for typical plasma at the L-H transition is depicted in figure 1 (left), with a very sharp density transition at the pedestal top producing a local minimum of the shear continuum $\omega_A^2(x)$, where $1/L_A = \omega_A^{2'}/\omega_A^2$ flips sign. Moreover, the finite pressure at the boundary sets a lower limit for the sound continuum $\omega_S^2(x)$. Because any frequency root above the sound-continuum (SC) minimum (local or absolute) yields a singular and thus damped wave, the diagram in figure 1 (right) provides the conditions for the existence of undamped oscillations in terms of the inhomogeneity parameter defined as $\Lambda = 1/(\lambda L_A K^2)$ and the related gradient $\omega_A^{2'} = \omega_A^2 \big/ L_A$:

- If $\{\Lambda, \omega_A^{2'}\} < 0$, then $\tilde{\omega}_\sigma^2 > 1$ lies in the SC (damped) and $\tilde{\omega}_\alpha^2 < 0$ (cannot oscillate);
- If $0 < \{\Lambda, \omega_A^{2'}\} < 1$, both $\{\tilde{\omega}_\alpha^2, \tilde{\omega}_\sigma^2\} > 1$ lie in the SC and are damped;
- Above $\Lambda_c = 1 + 2\sqrt{\beta} K_\parallel^2/K^2$ undamped oscillations are allowed, with $0 < \{\tilde{\omega}_\alpha^2, \tilde{\omega}_\sigma^2\} < 1$.

Near the critical inhomogeneity threshold Λ_c, the two roots of the normalised frequency are degenerate and the single value

$$\tilde{\omega}_c^2 = 1 \Big/ \Big(1 + \sqrt{\beta} K_\parallel^2/K^2\Big), \tag{15}$$

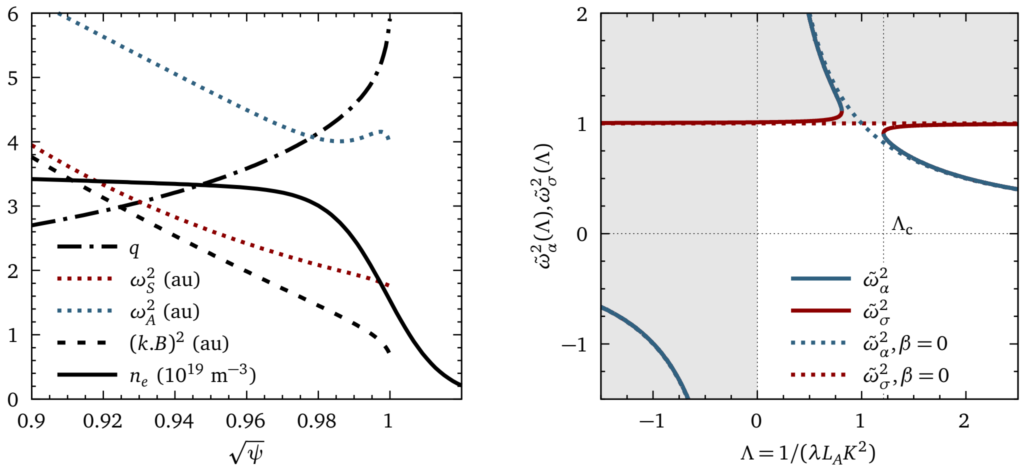

Figure 1: Left: radial profiles near the edge for a typical JET-like plasma at the L-H transition (black), along with the squared sound and shear continuum frequencies (colour). Right: squared frequency roots $\tilde{\omega}_\alpha^2, \tilde{\omega}_\sigma^2$ as functions of the inhomogeneity parameter Λ.

is independent of m if $n = 0$. Therefore, each $m > 1$ would yield a LFO with frequency at an integer multiple of the one for $m = 1$, eventually accounting for the regular frequency bands often observed [3]. However, while Λ_c is also independent of m for $n = 0$, the value $(\lambda L_A)_c$ is not. The degenerate frequency in (15) may also be written, in denormalised form, as

$$\omega_c^2 = \frac{1}{2}\Big[\omega_S^2(0) + \lambda L_A\big(k^2/\lambda^2\big)\omega_A^2(0)\Big], \tag{16}$$

evidencing a simultaneous scaling with $\omega_S^2(0)$ and $\omega_A^2(0)$.

In summary, a dispersion relation similar to the one of internal waves in neutral fluids was derived for MHD waves in inhomogeneous plasmas and slab geometry. It predicts oscillations with frequencies scaling with the poloidal Alfvén frequency but with values much lower than that of typical shear waves, thus providing a possible explanation for reported measurements of LFOs at the L-H transition in several fusion devices.

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References

[1] D. I. Réfy et al., Nucl. Fusion **60**, 056004 (2020).
[2] G. Birkenmeir et al., Nucl. Fusion **56**, 086009 (2016).
[3] E. R. Solano et al., Nucl. Fusion **57**, 022021 (2017).
[4] O. Grover et al., Nucl. Fusion **64**, 026001 (2024).
[5] D. J. Tritton, Physical Fluid Dynamics, Oxford University Press, (1998).
[6] K. Appert, R. Gruber, J. Vaclavik, Phys. Fluids **17**, 1471 (1974).