A Probabilistic Framework for Three-Dimensional Equilibrium Reconstruction

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Motivation

Magnetic equilibrium provides the magnetic-field geometry required to interpret measurements from multiple diagnostics and reconstruct local plasma quantities.

Deterministic reconstruction usually returns one best-fit equilibrium. For predictive use, however, we also need to quantify:

- diagnostic uncertainty,
- uncertainty in kinetic and current profiles,
- boundary and geometry uncertainty,
- degeneracy between profiles, geometry, and measurements.

Goal: infer a probability distribution of equilibria consistent with diagnostics and force balance, rather than a single deterministic state.

Main Contributions

- A unified Bayesian formulation for three-dimensional equilibrium reconstruction.
- Explicit coupling of flux-space latent quantities to real-space diagnostics.
- Probabilistic treatment of force balance, numerical truncation, and model discrepancy.
- Joint inference of geometry, kinetic profiles, and current-related flux functions with uncertainty propagation to diagnostics.

Core Idea

The framework keeps the distinction between two spaces:

Flux space	**Real space**
(ρ, θ, ζ)	(R, ϕ, Z)
latent geometry and profiles	detector locations
magnetic surfaces	measured signals
priors and force balance	diagnostic forward models

They are connected by the geometric map

$$\mathbf{X}(\rho, \theta, \zeta) = (R(\rho, \theta, \zeta), \phi = \zeta, Z(\rho, \theta, \zeta)).$$

An inverse coordinate transformation maps diagnostic locations back to flux coordinates.

The latent state contains the geometry coefficients, pressure and kinetic profiles, and either a toroidal-current or rotational-transform profile. Smoothness and boundary behavior are encoded through Gaussian-process priors.

Under stellarator symmetry, the geometry is represented spectrally,

$$R = \sum_{m,n} R_{mn}(\rho) \cos(m\theta - n\zeta), \qquad Z = \sum_{m,n} Z_{mn}(\rho) \sin(m\theta - n\zeta),$$

with the radial coefficient functions treated as smooth latent functions.

Boundary Treatment with Gaussian Processes

Latent radial functions are modeled as

$$f(\rho) \sim \mathcal{GP}(m(\rho), k(\rho, \rho')), \qquad \mathcal{L}_b f = b,$$

so boundary values or derivatives are imposed by GP conditioning.

At the magnetic axis $\rho = 0$, regularity requires

$$R_{mn}(0) = Z_{mn}(0) = 0, \qquad m \neq 0,$$

and

$$\lim_{\rho \to 0} \lambda_\theta(\rho, \theta, \zeta) = \lim_{\rho \to 0} \left[\frac{\sqrt{g}/g_{\zeta\zeta}}{\langle \sqrt{g}/g_{\zeta\zeta} \rangle} - 1 \right] = 0, \qquad m \neq 0.$$

The normalized poloidal average needed for this limit is the minor correction to the original VMEC presentation.

Force Balance as a Virtual Observation

Instead of enforcing ideal-MHD force balance as an exact hard constraint,

$$\mathbf{J} \times \mathbf{B} - \nabla p = 0,$$

we treat the residual probabilistically:

$$\mathbf{F}_{\rm eq} = \mathbf{J} \times \mathbf{B} - \nabla p \sim \mathcal{N}(0, \Sigma_{\rm eq}).$$

Interpretation: force balance is enforced with a tolerance that accounts for spectral truncation, discretization error, and model discrepancy. This connects weak-form VMEC-like consistency with pointwise DESC-like residual control within one probabilistic model.

Probabilistic Framework

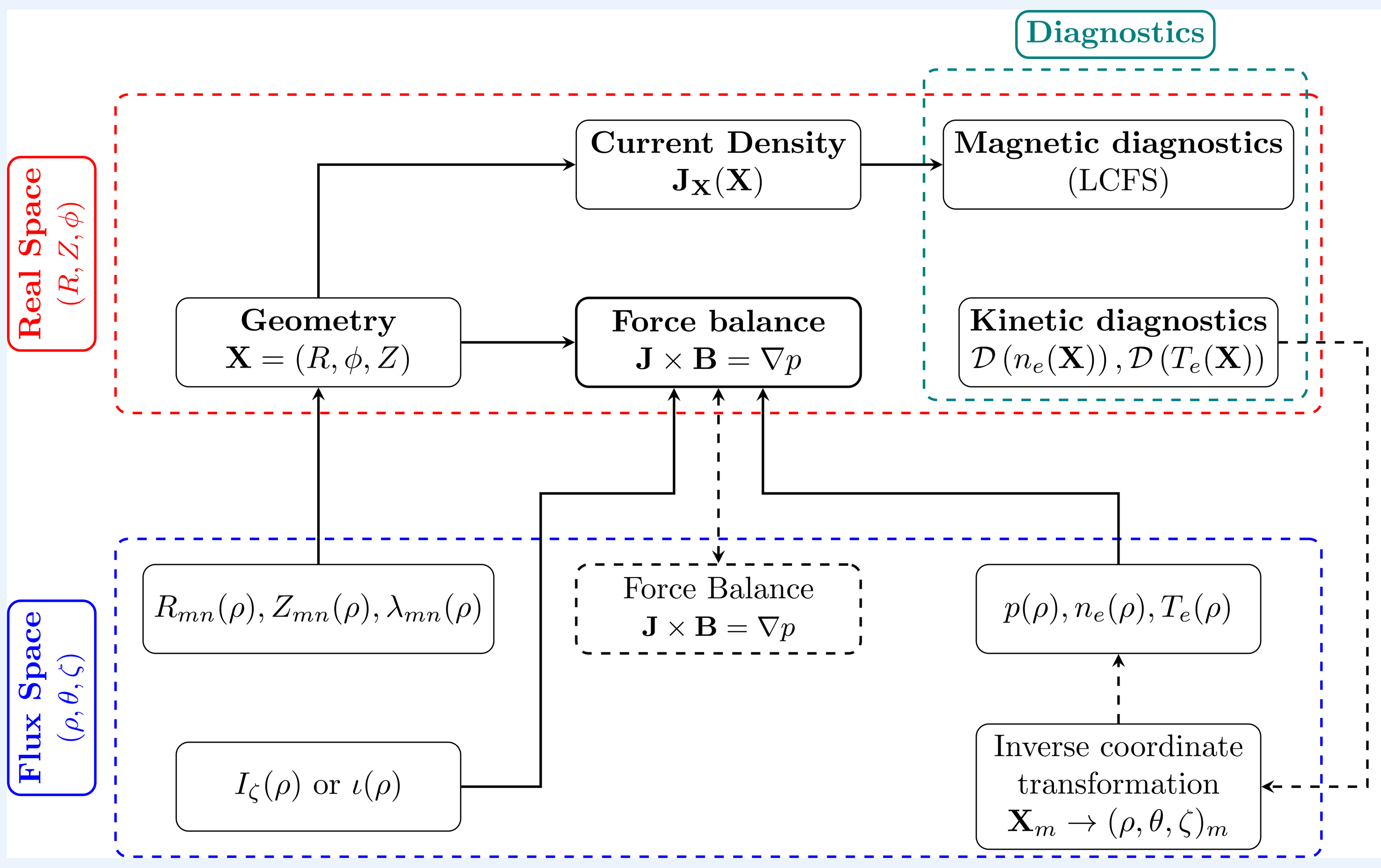

Figure 1. Bayesian equilibrium reconstruction framework. Latent geometry, kinetic profiles, and current-related quantities are defined in flux space, while diagnostic observations are evaluated in real space through forward models and inverse coordinate transformations.

The posterior combines prior information, diagnostic measurements, and a probabilistic force-balance constraint:

$$p(\xi \mid \mathcal{D}) \propto p(\xi)\, p(\mathcal{D}_{\rm kin} \mid \xi)\, p(\mathcal{D}_{\rm mag} \mid \xi)\, p_{\rm eq}(\xi).$$

Here ξ denotes the latent equilibrium state: geometry, kinetic profiles, current-related profiles, and boundary variables.

Inference Structure

- **Latent state:** Fourier geometry coefficients, kinetic profiles, and current-related flux functions.
- **Diagnostics:** magnetic probes, line-integrated density, Thomson scattering, and ECE-like constraints.
- **Forward models:** real-space diagnostic locations are mapped to flux coordinates before profile evaluation.
- **Physics:** ideal-MHD force balance is treated as a virtual observation.

For local kinetic measurements, $\mathbf{X}_m \mapsto (\rho_m, \theta_m, \zeta_m)$ and $D_m = H_m(\xi) + \epsilon_m$, with $\epsilon_m \sim \mathcal{N}(0, \sigma_m^2)$.

Synthetic Diagnostic System

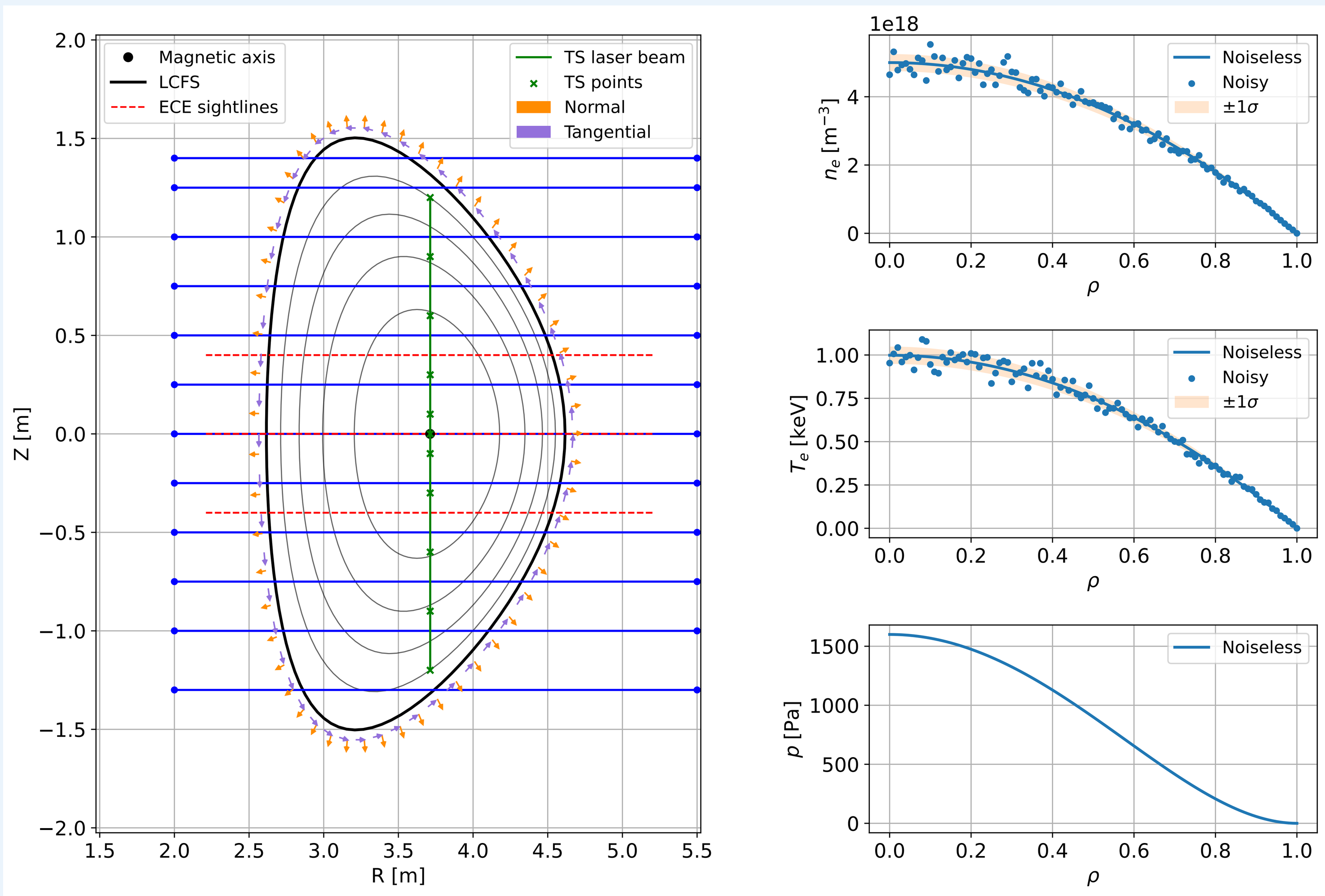

Figure 2. Synthetic diagnostic layout for validation. The controlled system provides magnetic, interferometer, Thomson-scattering, and ECE-like observations from a known reference equilibrium.

Synthetic diagnostics generate controlled observations from a known reference equilibrium. This makes it possible to test whether the posterior recovers the correct plasma state and uncertainty level.

Representative signals include magnetic pick-up coils, line-integrated density, Thomson-scattering points, ECE sightlines, and boundary information. Controlled noise realizations test posterior coverage, identifiability, and uncertainty calibration.

Validation Strategy and Expected Outcome

Current milestone: the 2D synthetic diagnostic system is nearly complete, and full posterior-recovery validation is close.

1. 2D validation
Validate the framework against a known axisymmetric reference equilibrium.

2. 3D validation
Extend to non-axisymmetric VMEC equilibria and fully 3D forward models.

3. Experimental data
Apply the validated framework to real measurements with unknown ground truth.

Outcome: a posterior distribution of physically plausible equilibria with quantified uncertainty.

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This work has been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No. 101052200 — EUROfusion). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.