New formulation for plasma fluids with exact discrete conservation in arbitrary curvilinear geometry

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Introduction

General curvilinear coordinates are an essential tool in plasma modeling [1], since they help represent magnetic geometry, dynamic anisotropy, and localized fine scales. They also help preserve natural symmetries of the problem. The usual conservative or advective formulations of fluids require explicit geometric source terms introduced through the covariant derivatives. For example, in Eulerian formulation the Reynolds stress reads $[\nabla\cdot(\rho\mathbf{v}\mathbf{v})]_i = \frac{1}{\mathscr{J}}\partial_j\left(\mathscr{J}\rho v_i v^j\right) - \Gamma^k_{ij}\rho v^j v_k$, where the second term must be retained to recover the correct fictitious forces and conservation properties. In numerical applications, geometry is commonly handled using bespoke numerical schemes, usually finite element or finite volume methods specifically adapted to the underlying problem in cylindrical or spherical geometry.

This paper describes a new formulation of fluids that retains rigorous geometry effects while avoiding explicit Christoffel-symbol sources. Furthermore, we formulate the plasma equations in a form that easily remains conserving when represented in a discrete setting. Although the proofs are presented –for clarity– using continuous operators, the arguments carry over verbatim to the discrete case. In what follows, we use standard notation [1], and focus on the representation and its conservative structure. A complete discussion can be found in [2].

New curvilinear formulation of plasmas

The central idea is to evolve Jacobian-weighted variables for which mass, kinetic energy, and internal energy are Euclidean L^2 invariants in a curvilinear space with coordinates ξ^i. We define generalized fluid moments $\underline{\rho} \equiv \sqrt{\mathscr{J}\rho}$, $\underline{\mathbf{m}} \equiv \sqrt{\mathscr{J}\rho}\,\mathbf{v}$, and $\underline{u} \equiv \sqrt{2\mathscr{J}u}$, where $u = p/(\gamma-1)$. Then, the total mass, kinetic energy, and internal energy are

$$\int \rho\, dV = \int \underline{\rho}^2\, d^3\xi, \qquad \int \frac{1}{2}\rho v^2\, dV = \int \frac{1}{2}|\underline{\mathbf{m}}|^2\, d^3\xi, \qquad \int u\, dV = \int \frac{1}{2}\underline{u}^2\, d^3\xi. \tag{1}$$

Additionally, we introduce the Jacobian-weighted divergence

$$\nabla^\dagger\cdot\mathbf{A} \equiv \mathscr{J}\nabla\cdot\left(\frac{\mathbf{A}}{\mathscr{J}}\right), \tag{2}$$

which is adjoint to the directional derivative $\mathbf{A}\cdot\nabla$ under Euclidean space inner product. Furthermore, we derive expressions for the Reynolds stress and the viscosity that avoid the covariant

derivative. These choices conceal geometric dependencies inside the variables and prevent non-linear source terms from appearing in the equations.

As an example, we apply our new approach to the visco-resistive MHD model. This is a representative system containing the relevant flow, viscous, force, and resistive operators appearing in the fluid hierarchy. Using the new variables the equations read

$$\partial_t \underline{\rho} + \frac{1}{2}\left(\nabla^\dagger \cdot \mathbf{v} + \mathbf{v}\cdot\nabla\right)\underline{\rho} = 0, \tag{3}$$

$$\begin{aligned}\partial_t \underline{\mathbf{m}} &+ \frac{1}{2}\left(\mathbf{v}\nabla^\dagger + \nabla\mathbf{v}\right)\cdot\underline{\mathbf{m}} + \frac{1}{2}\left(\mathbf{v}\mathbf{v} - \mathbf{I}v^2\right)\cdot\nabla\underline{\rho} - \mathbf{v}\times\nabla\times\underline{\mathbf{m}} \\ &+ \frac{\gamma-1}{\underline{\rho}}\left(\underline{u}\nabla\underline{u} - \underline{u}^2\mathbf{G}\right) + \mathscr{J}\frac{\mathbf{B}}{\mu_0\underline{\rho}}\times\nabla\times\mathbf{B} + \frac{\mathscr{J}}{\underline{\rho}}\nabla\cdot\tau = 0,\end{aligned} \tag{4}$$

$$\begin{aligned}\partial_t \underline{u} &+ \frac{\gamma}{2}\left(\nabla^\dagger\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\right)\underline{u} - (\gamma-1)\mathbf{v}\cdot(\nabla\underline{u} - \underline{u}\mathbf{G}) \\ &+ \frac{\mathscr{J}}{\underline{u}}\left[\nabla\cdot(\mathbf{v}\cdot\tau) - \mathbf{v}\cdot\nabla\cdot\tau\right] - \frac{\mathscr{J}}{\underline{u}}\left(\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla\right)\times\left(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}\right) = 0,\end{aligned} \tag{5}$$

$$\partial_t\mathbf{B} - \nabla\times\left(\mathbf{v}\times\mathbf{B} - \frac{\eta}{\mu_0}\nabla\times\mathbf{B}\right) = 0. \tag{6}$$

Here $\mathbf{G} = (\sqrt{\mathscr{J}}\nabla\sqrt{\mathscr{J}})/\mathscr{J}$. The Ohmic heating term has been written in a manifestly symmetric form equivalent to ηJ^2 in the continuum, but better suited to discrete energy conservation. The viscosity tensor is defined as $\tau = -\mu\left[-\mathbf{I}\times\nabla\times\mathbf{v} + \alpha\mathbf{I}(\nabla\cdot\mathbf{v})\right]$, which avoids explicit covariant derivatives while retaining the continuum vector-Laplacian structure. This is not intended as a replacement for the full-Braginskii viscous stress [3]; it is a simple isotropic viscous operator useful for numerical regularization and verification.

Mass, momentum, and energy conservation properties

Mass conservation is obtained by multiplying Eq. (3) by $\underline{\rho}$ and integrating over volume:

$$\frac{\partial}{\partial t}\int \rho\, dV = \frac{\partial}{\partial t}\int \underline{\rho}^2\, d^3\xi = -\int \underline{\rho}\left(\nabla^\dagger\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\right)\underline{\rho}\, d^3\xi = 0. \tag{7}$$

Thus mass conservation requires only the anti-symmetry of the first-derivative operator and appropriate boundary conditions.

Individual momentum components are generally not conserved in curvilinear coordinates because non-ignorable coordinates give rise to geometric source terms. In the special case where ξ^i is cyclic, so that the metric and Jacobian are independent of ξ^i, the corresponding covariant momenta are conserved:

$$\frac{\partial}{\partial t}\int \rho v_i\, dV = 0, \;\text{ if }\; \partial_i\mathscr{J} = 0 \;\text{ and }\; \partial_i g^{jk} = 0. \tag{8}$$

This includes angular momentum conservation in polar or toroidal-like ignorable directions. Cartesian momentum conservation is obtained as a special case of the above. When coordinates

are not ignorable, the metric-dependent terms reproduce the usual fictitious forces, however obtained without evaluating Christoffel symbols.

The energy conservation relation is obtained by dotting Eq. (4) with $\underline{\mathbf{m}}$, multiplying Eq. (5) by $\underline{u}$, and dotting Eq. (6) with $\mathscr{J}\mathbf{B}/\mu_0$. The terms $\underline{\mathbf{m}}\cdot(\mathbf{vv}-\mathbf{I}v^2)\cdot\nabla\underline{\rho}=0$ and $\underline{\mathbf{m}}\cdot(\mathbf{v}\times\nabla\times\underline{\mathbf{m}})=0$ vanish by orthogonality. After evaluating the remaining terms, we recover the total energy

$$\frac{\partial}{\partial t}\int\left(\frac{1}{2}\rho v^2+u+\frac{B^2}{2\mu_0}\right)dV=0. \tag{9}$$

Therefore, Equations (3)–(6) conserve total mass and energy in arbitrary curvilinear coordinates, and recover the expected momentum or angular-momentum invariants in the corresponding Cartesian or cyclic coordinate limits.

Testing energy conservation in non-orthogonal curvilinear coordinates

The Orszag–Tang vortex [4] is commonly utilized as a nonlinear MHD test because the initially smooth, doubly periodic fields rapidly generate interacting waves, current sheets, shocks, and steep gradients. The problem is posed in a Cartesian unit square. Here we evolve it in the distorted periodic coordinates

$$x(\chi,\zeta)=\chi+a\sin\left(k_\zeta\,\zeta\right) \tag{10}$$

$$y(\chi,\zeta)=\zeta+b\sin\left(k_\chi\,\chi\right) \tag{11}$$

with $a=b=0.05$ and $k_\chi=k_\zeta=2$. This mapping produces a non-orthogonal metric and a spatially varying Jacobian $J(\chi,\zeta)=1-abk_\chi k_\zeta\cos(k_\chi\chi)\cos(k_\zeta\zeta)$, so that the calculation exercises the full curvilinear formulation. The initial conditions are prescribed in physical Cartesian space and match the standard Orszag–Tang parameters. The new curvilinear formulation is implemented using **ALMA** [5].

Figure 1 show the density obtained with the curvilinear mapping compared against the Cartesian result at $t=0.5\,\tau_A$. The dominant nonlinear structures are preserved: compression develops near the central stagnation region, the density gradients sharpen, and the large-scale morphology remains close to the Cartesian reference solution.

A key result is that the calculation preserves the structural invariants of the formulation. The magnetic field remains solenoidal, and the volume-integrated total energy is conserved to approximately eleven digits of accuracy when using an implicit midpoint time integrator. The test demonstrates that the proposed curvilinear representation can evolve a nonlinear MHD problem in non-orthogonal coordinates while retaining exact discrete energy conservation. Overall, the comparison showcases the role of underlying physical symmetries in the plasma dynamics, and validates the approach in general curvilinear geometries.

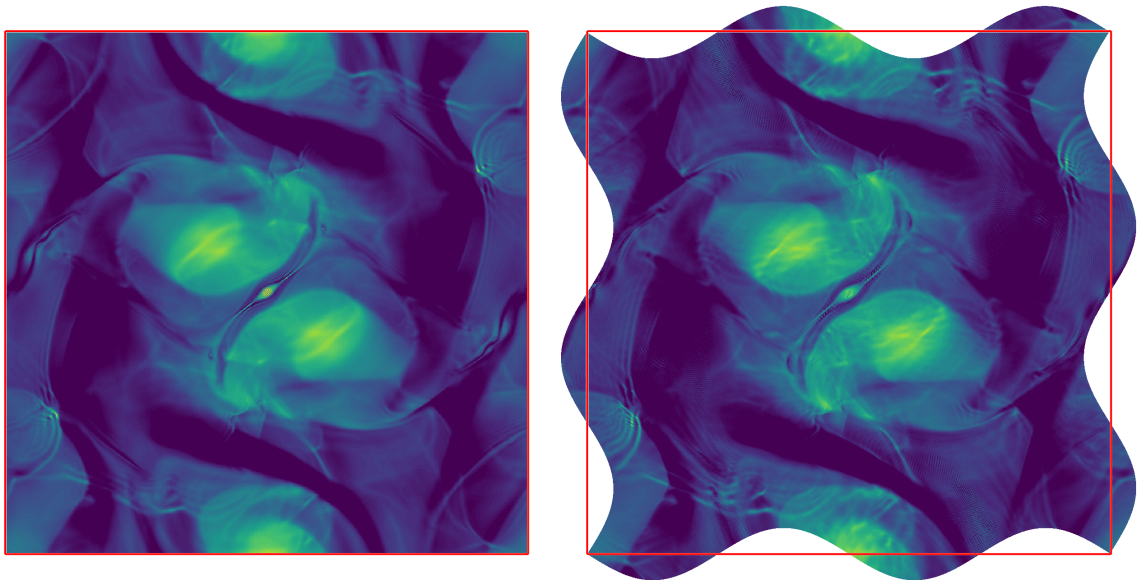

Figure 1: Plasma density at $t = 0.5\tau_A$ for simulations of the Orzsag-Tang vortex problem using Cartesian (left) or curvilinear coordinates (right).

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