New formulation for plasma fluids with exact discrete conservation in arbitrary curvilinear geometry

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Motivation

General curvilinear coordinates are an essential tool for plasma simulations. Standard Eulerian or Lagrangian curvilinear formulations introduce quadratic non-linear source terms $\sim \Gamma^k_{ij}\rho v^j v_k$ (fictitious forces) that are challenging to implement and to discretize. We formulate a new representation of plasmas that rigorously retains all geometry effects by absorbing the metric derivatives into the flow operators. Notably, the new representation is constructed such that the main conservation laws follow from operator antisymmetry and product orthogonality – which directly translates to discrete space

1. New formulation of fluids in curvilinear coordinates

- **Core idea:** construct variables and fluxes $\Rightarrow$ conservation from antisymmetry and orthogonality

$$\varrho \equiv \sqrt{\mathcal{J}\rho}, \qquad \mathbf{m} \equiv \sqrt{\mathcal{J}\rho}\,\mathbf{v}, \qquad \underline{u} \equiv \sqrt{2\mathcal{J}u}, \qquad \text{where } u = p/(\gamma-1)$$

- The total mass, kinetic energy, and internal energy are quadratic quantities

$$\int \rho\, dV = \int \varrho^2\, d^3\xi, \qquad \int \frac{1}{2}\rho v^2\, dV = \int \frac{1}{2}|\mathbf{m}|^2\, d^3\xi, \qquad \int u\, dV = \int \frac{1}{2}\underline{u}^2\, d^3\xi$$

- Use a Jacobian-weighted divergence adjoint to the directional derivative $\mathbf{A}\cdot\nabla$

$$\nabla^\dagger\cdot\mathbf{A} \equiv \mathcal{J}\nabla\cdot\left(\frac{\mathbf{A}}{\mathcal{J}}\right) \Rightarrow \int f\left(\nabla^\dagger\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\right) f\, d^3\xi = 0$$

- Formulate the model using anti-symmetric and orthogonal flow operators

$$\partial_t\varrho + \frac{1}{2}\left(\nabla^\dagger\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\right)\varrho = 0$$

$$\partial_t\mathbf{m} + \frac{1}{2}\left(\mathbf{v}\nabla^\dagger + \nabla\mathbf{v}\right)\cdot\mathbf{m} + \frac{1}{2}\left(\mathbf{v}\mathbf{v} - \mathbf{I}v^2\right)\cdot\nabla\varrho - \mathbf{v}\times\nabla\times\mathbf{m}$$
$$+ \frac{\gamma-1}{\varrho}\left(\underline{u}\nabla\underline{u} - \underline{u}^2\mathbf{G}\right) + \mathcal{J}\frac{\mathbf{B}}{\mu_0\varrho}\times\nabla\times\mathbf{B} + \frac{\mathcal{J}}{\varrho}\nabla\cdot\tau = 0, \qquad \tau = -\mu\left[-\mathbf{I}\times\nabla\times\mathbf{v} + \alpha\mathbf{I}(\nabla\cdot\mathbf{v})\right]$$

$$\partial_t\underline{u} + \frac{\gamma}{2}\left(\nabla^\dagger\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\right)\underline{u} - (\gamma-1)\mathbf{v}\cdot\left(\nabla\underline{u} - \underline{u}\mathbf{G}\right)$$
$$+ \frac{\mathcal{J}}{\underline{u}}\left[\nabla\cdot(\mathbf{v}\cdot\tau) - \mathbf{v}\cdot\nabla\cdot\tau\right] - \frac{\mathcal{J}}{\underline{u}}\left(\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla\right)\times\left(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}\right) = 0$$

$$\partial_t\mathbf{B} - \nabla\times\left(\mathbf{v}\times\mathbf{B} - \frac{\eta}{\mu_0}\nabla\times\mathbf{B}\right) = 0$$

2. Conservation properties

- **Core idea:** Continuous conservation laws satisfied in discrete space by requiring
  - **Anti-symmetry** of first derivative $\mathbf{D} = -\mathbf{D}^T$
  - **Orthogonality** of inner and cross products
  - **Solenoidal** magnetic field $\nabla\cdot\mathbf{B} = 0$
- **Mass conservation** uses anti-symmetry of curvilinear scalar flow operator

$$\frac{\partial}{\partial t}\int \rho\, dV = \frac{\partial}{\partial t}\int \varrho^2\, d^3\xi = -\int \varrho\left(\nabla^\dagger\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\right)\varrho\, d^3\xi = 0$$

- **Momentum conservation** of individual components only in cyclic or ignorable coordinates

$$\frac{\partial}{\partial t}\int \rho v_i\, dV = 0, \quad \text{if } \partial_i\mathcal{J} = 0, \quad \partial_i g^{jk} = 0, \quad \text{and} \quad \nabla\cdot\mathbf{B} = 0$$

  including angular momentum in polar or toroidal coordinates – otherwise: fictitious forces
- **Energy conservation** arises from anti-symmetry and orthogonality

$$\int \underline{u}\left(\nabla^\dagger\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\right)\underline{u}\, d^3\xi = 0 \qquad \text{Thermal flux}$$

$$\int \mathbf{m}\cdot\left[\underbrace{\frac{1}{2}\left(\mathbf{v}\nabla^\dagger + \nabla\mathbf{v}\right)\cdot\mathbf{m}}_{\text{Anti-symmetry}} + \underbrace{\frac{1}{2}\left(\mathbf{v}\mathbf{v} - \mathbf{I}v^2\right)\cdot\nabla\varrho - \mathbf{v}\times\nabla\times\mathbf{m}}_{\text{Orthogonality}}\right] d^3\xi = 0 \qquad \text{Kinetic flux}$$

$$\int \frac{\mathcal{J}}{\mu_0}\underbrace{\mathbf{B}\cdot\left[\nabla\times(\mathbf{v}\times\mathbf{B}) + \mathbf{v}\times\nabla\times\mathbf{B}\right]}_{\text{Anti-symmetry}} d^3\xi = 0 \qquad \text{Ideal Poynting flux}$$

Total sum is conserved $\partial_t\int dV\left(\frac{1}{2}\rho v^2 + \frac{B^2}{\mu_0} + u\right) = 0$

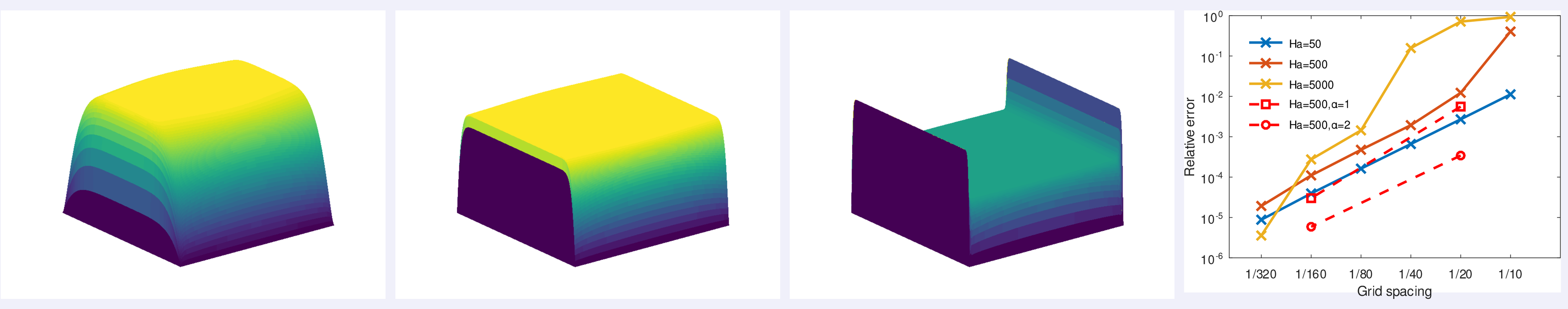
3. Steady-state inductive Shercliff flows

- **Test viscous and resistive operators** using a conducting fluid through a square channel with

$$\mathbf{B}(t=0) = B_0\mathbf{y}, \qquad \mathbf{v}|_{\text{wall}} = 0, \qquad (B_x, B_y, B_z)|_{\text{wall}} = (0, B_0, 0).$$

- Flow characterized by $Ha = B_0 L\sqrt{\frac{1}{\mu\rho\eta}}$, thin MHD boundary layers $\delta_H/L \sim Ha^{-1}$, $\delta_S/L \sim Ha^{-1/2}$
- Compare Cartesian geometry results with curvilinear calculation using compression near wall

$$x(\chi) = \chi_0 + \frac{1}{2}L_x\left\{1 + \frac{\tanh\left[\alpha_x\left(1 - 2\frac{\chi-\chi_m}{L_x}\right)\right]}{\tanh\alpha_x}\right\}$$



Left: Axial velocity profiles at $Ha = 50, 500, 5000$. Right: Second order spacial accuracy recovered with new representation

- **Takeaway:** Integrated flow rate $Q = \int v_z\, dV$ agrees with the analytical Shercliff solution
- Smooth curvilinear wall compression improves accuracy by $\sim 10\times$ at fixed grid size

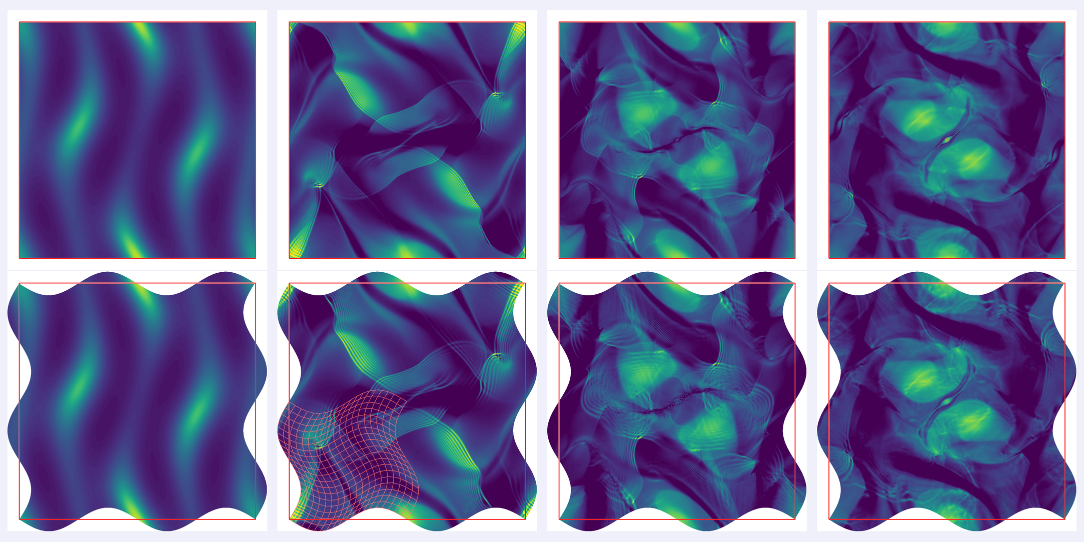
4. Orszag–Tang vortex in non-orthogonal coordinates

- **Test energy conservation** with non-linear MHD problem using the Orszag–Tang vortex

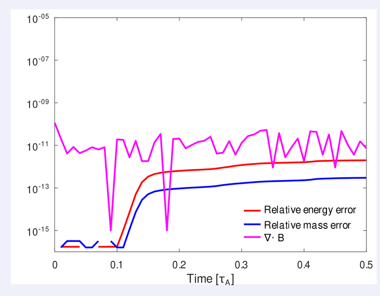
$$\rho(t=0) = \gamma^2/(4\pi) \qquad \mathbf{v}(t=0) = -\sin(2\pi y)\mathbf{x} + \sin(2\pi x)\mathbf{y}$$
$$p(t=0) = \gamma/(4\pi) \qquad \mathbf{B}(t=0) = -B_0\sin(2\pi y)\mathbf{x} + B_0\sin(4\pi x)\mathbf{y} \qquad \gamma = 5/3, \quad B_0 = 1/\sqrt{4\pi}.$$

- Compare Cartesian evolution with a non-orthogonal curvilinear calculation using

$$x(\chi,\zeta) = \chi + a\sin(k_\zeta\zeta), \qquad y(\chi,\zeta) = \zeta + b\sin(k_\chi\chi),$$



Density evolution in Cartesian and curvilinear coordinates



Total energy error using implicit midpoint

- **Takeaway:** energy conserved to 11 digits and $\nabla\cdot\mathbf{B} = 0$ is retained in non-orthogonal coordinates

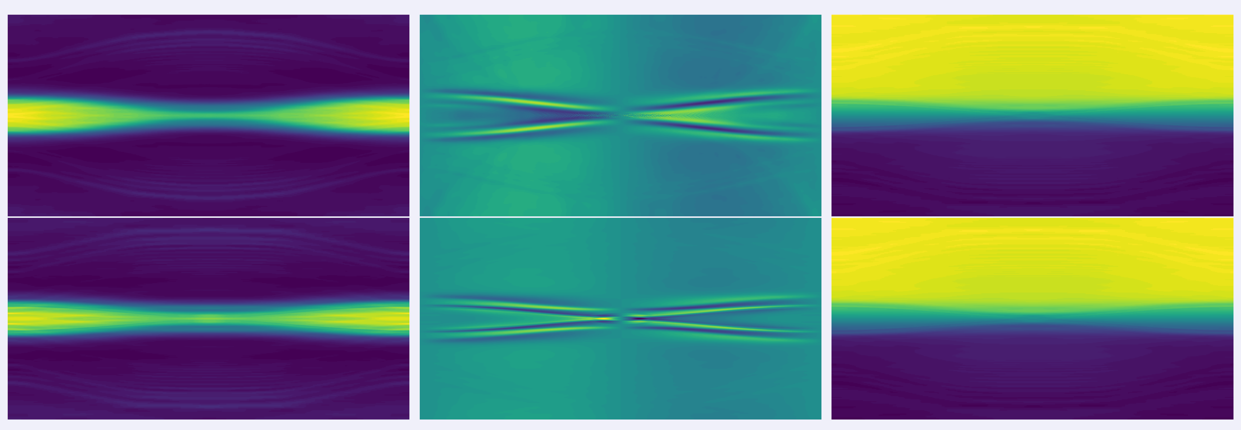
5. GEM reconnection problem

- Start with perturbed Harris sheet with a trigger perturbation

$$\mathbf{B}(t=0) = B_0\tanh(y/\lambda)\mathbf{x}, \qquad n(t=0) = n_b + \frac{n_s}{\cosh^2(y/\lambda)}$$
$$\delta A_z = A_0\cos(2\pi x/L_x)\cos(\pi y/L_y)$$

- **Test grid compression** around reconnection layer to resolve X-point

$$y(\xi) = \xi - \frac{4}{(\xi_1-\xi_0)^2}\left(1 - \frac{1}{C}\right)(\xi-\xi_0)\left(\xi - \frac{\xi_0+\xi_1}{2}\right)(\xi_1-\xi), \qquad \left.\frac{dy}{d\xi}\right|_{\xi=(\xi_0+\xi_1)/2} = \frac{1}{C}.$$



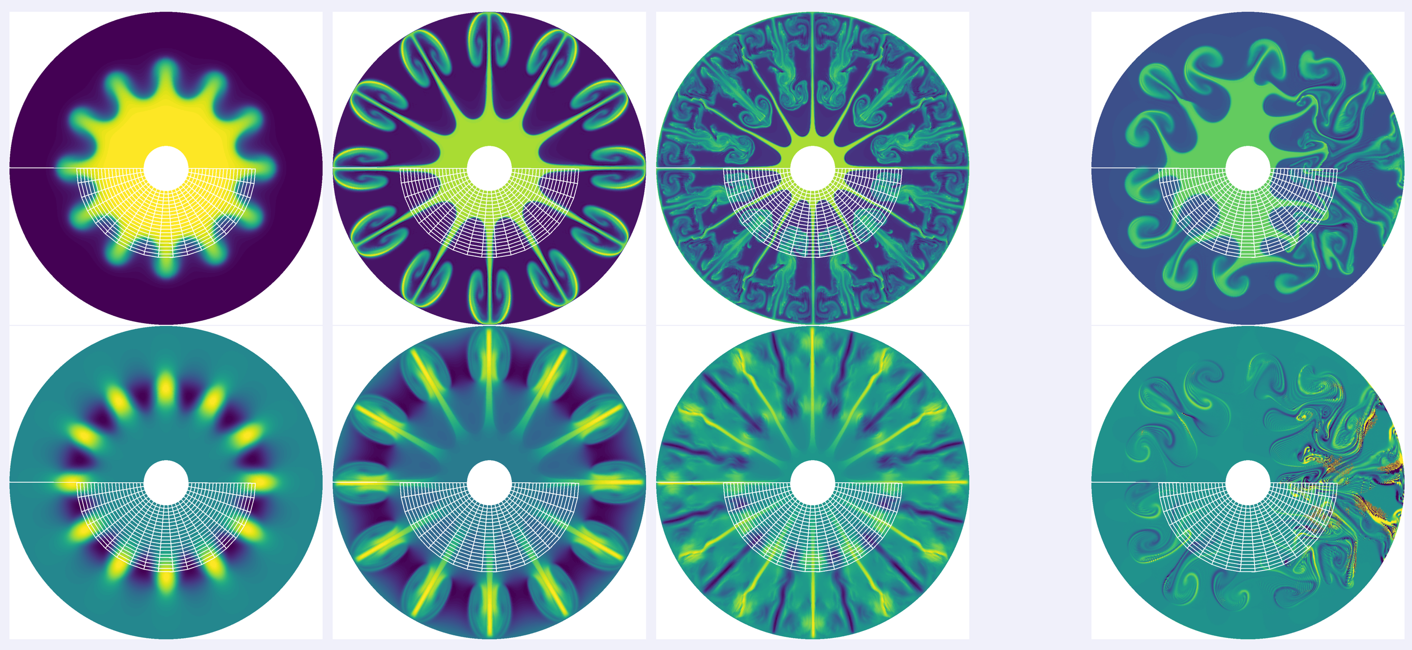
Calculated ρ (left), v_x (center), B_x (right) profiles at $S = 10^3$ (top) $S = 10^7$ (bottom) with $C_y = 4$

- **Takeaway:** targeted coordinate compression resolves the current sheet dynamics
- Calculation reproduces the current-sheet thinning at high Lundquist number

6. Magnetized Rayleigh–Taylor instability in polar coordinates

- **Gravity-driven instability** of a heavy fluid supported against a light fluid
- Test is performed in polar coordinates, angular symmetry is sensitive to curvilinear metric
- The density is initialized with a smooth hyperbolic-tangent interface

$$\rho(t=0) = \rho_{\text{avg}} - \Delta\rho\tanh[(r-r_0)/\delta], \qquad v_r = A\sin(m\theta)\exp[-(r-r_0)^2/\sigma^2]$$



Left: ρ (top) and v_r (bottom) evolution with $m = 12$. Right: ρ and induced B_z but initializing with superimposed modes

- **Takeaway:** Problem symmetry retained through ang. momentum conservation in polar angle

Conclusions

- **New curvilinear representation supports very different plasma-fluid problems:** liquid-metal flows, magnetic reconnection, ICF-like interface dynamics, turbulence
- **The representation preserves physical structure:** mass, total energy, and momenta associated with ignorable coordinates follow from antisymmetric operators, product orthogonality, and $\nabla\cdot\mathbf{B} = 0$.
- **The tests verify different aspects of the formulation:** Shercliff flow verifies the physical viscosity and resistive operators against analytical/benchmark results; Orszag–Tang verifies nonlinear energy conservation in non-orthogonal coordinates; MRT tests poloidal symmetry and force balance
- **Mapped coordinates improve resolution without changing the model:** grid compression resolves Hartmann/Shercliff layers and GEM current sheets using the same conservative machinery
- **Geometry, conservation, and resolution control are handled at the representation level** avoiding specialized equation sets or geometry-specific source-term implementations for each application

Details on the model and conservation proofs are presented in Phys. Plasmas 33, 053904 (2026). The authors would like to thank R.E. Waltz and O. Koshkarov. *This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Sciences, Theory Program, under Award DE-FG02-95ER54309.*

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