Three-dimensional steady states as perturbations of the Solov'ev equilibrium

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Motivation and Aim

Magnetic confinement of laboratory fusion plasmas involves the existence of well defined closed and nested toroidal magnetic surfaces. For two-dimensional (2D) magnetohydrodynamic (MHD) equilibria, e.g. in the presence of axisymmetry, the existence of such surfaces is rigorously guaranteed. However, in the absence of any continuous spatial symmetry the existence of steady states with nested toroidal magnetic surfaces has been questioned because the breaking of symmetry allows for magnetic field-line braiding [1]. For this reason, in order to obtain steady states with favourable confinement properties, particularly in the framework of stellator optimization, some kinds of symmetry have been imposed, e.g. quasisymmetry in which B having a continuous symmetry in certain coordinate systems becomes independent of one of the coordinates [2]. Recently, weakly asymmetric plasma equilibria exhibiting nested toroidal magnetic surfaces and isotropic pressure were constructed without any symmetry assumption, thus indicating counter examples to the aforementioned conjecture of non-existence of such 3D equilibria [3].

Aim of the present study is the construction of three-dimensional (3D) toroidal equilibria with pressure anisotropy, showcasing closed and nested magnetic surfaces on the basis of two foundations: first, a special class of equilibria with anisotropic pressure components given by Eqs. (4) valid for any (asymmetric) magnetic field, which was identified for the first time in [4]; and second, by introducing the representation (6) for the magnetic field in cylindrical coordinates (r, ϕ, z).

Axisymmetric Solov'ev equilibrium

Dimensionless Grad-Shafranov equation for axisymmetric MHD equilibria:

$$\Delta^\star\psi + F\frac{dF}{d\psi} + r^2\frac{dP}{d\psi} = 0\,, \quad \Delta^\star := \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r}\frac{\partial}{\partial r}, \tag{1}$$

where (r, ϕ, z) are cylindrical coordinates and $\psi(r, z)$ is the poloidal magnetic flux function.
Choice of the free functions: $P(\psi) = P_a - 2(1+\delta^2)\psi$ and $F(\psi) = (F_0^2 + 4\epsilon\psi)^{1/2}$.

The resulting equation admits the Solov'ev solution: $\psi_{ax} = z^2(r^2 - \epsilon) + \frac{\delta^2}{4}(r^2 - 1)^2.$ (2)

The respective up-down symmetric equilibrium is diamagnetic for $\epsilon \geq 0$ and paramagnetic for $\epsilon < 0$. In the present study we restrict ourselves to the diamagnetic configuration, shown in Fig. 1-left.

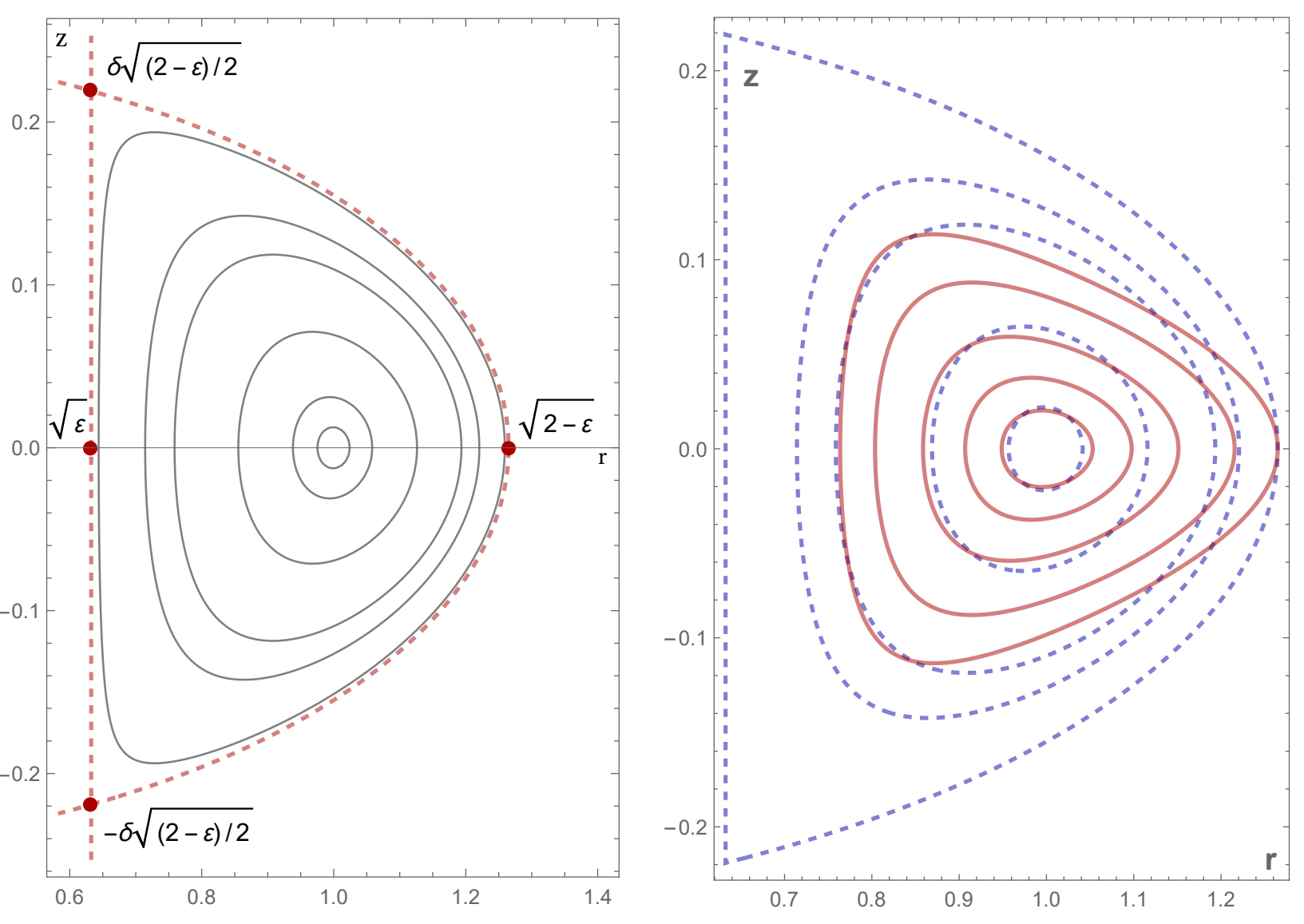

Figure 1: Left panel: The axisymmetric diamagnetic Solov'ev configuration for $\delta = \epsilon = 0.4$. Right panel: Respective isomagnetic contours (red-continuous curves) with the outer curve touching the separatrix. The blue-dashed curves represent the respective intersections of the magnetic surfaces with the poloidal plane.

It forms spontaneously a separatrix consisting of an outer elliptic part and an inner part parallel to the axis of symmetry, thus having a couple of X-points. The magnetic axis is located at $(z = 0,\ r = 1)$ on which $\psi_{ax} = 0$. For $\epsilon > 0$ the solution describes a tokamak-type equilibrium, while for $\epsilon = 0$ reduces to a spheromak one. Since the plasma boundary is not imposed from the outset, it can be chosen a posteriori to coincide with the separatrix or with one of the internal, smoothly closed magnetic surfaces.

The equilibrium has nested surfaces of constant magnetic field modulus (isomagnetic surfaces) without forming a separatrix with the isomagnetic axis located at $(z = 0, r = r^B_{ax})$; the radial coordinate, r^B_{ax}, can be obtained analytically even in the absence of symmetry in terms of the free parameters δ, ϵ and F_0. r^B_{ax} strongly increases with F_0 and weakly decreases with δ and ϵ. Therefore, pending on the values of the free parameters, the position of the isomagnetic axis can be either inside or outside the magnetic separatrix. An example of the former case is given in Fig. 1-right.

The framework of 3D equilibria

Governing equations for 3D equilibria with anisotropic pressure:

$$\nabla\cdot\boldsymbol{\mathcal{P}} + \mathbf{B}\times\mathbf{j} = 0\,, \quad \nabla\times\mathbf{B} = \mathbf{j}\,, \quad \nabla\cdot\mathbf{B} = 0\,, \tag{3}$$

where $\boldsymbol{\mathcal{P}} = P_\perp\boldsymbol{\mathcal{I}} + (P_\parallel - P_\perp)/B^2\mathbf{BB}$ is the pressure tensor consisting of one element representing the parallel to the magnetic field pressure component $(P_\parallel)$ and two equal perpendicular components $(P_\perp)$ with $\mathcal{I}$ the identity tensor.
Special class of solutions with constant pressure, P_0, not representing confined plasmas, valid for any magnetic field [4]:

$$P_\perp = P_0 - \frac{1}{2}B^2\,, \quad P_\parallel = P_0 + \frac{1}{2}B^2\,. \tag{4}$$

Identically divergence-free magnetic-field representation:

$$\mathbf{B} = \nabla(\phi + w(\phi))\times\nabla U(r,\phi,z) + I(r,z)\nabla\phi\,; \tag{5}$$

Specification of the free functions in (5):

$$U(r,\phi,z) = z^2[r^2 - \epsilon(1+g(\phi))] + \frac{\delta^2(1+h(\phi))}{4}(r^2-1)^2. \tag{6}$$

$$s(\phi) = c_s\cos(m_s\phi) + d_s\sin(n_s\phi) \quad (s = g, h, w)\,, \quad I(r,z) = (F_0^2 + 4\epsilon\psi_{ax})^{1/2}, \tag{7}$$

with ψ_{ax} given by (2).
To examine whether the equilibria determined by Eqs. (4)-(7) can have closed and nested magnetic surfaces we have traced out the magnetic field lines on the basis of the equation $d\mathbf{x}(l)/dl = \mathbf{B}(\mathbf{x})$, where l is the arc-length associated with the vector $\mathbf{x}$ tangential to $\mathbf{B}$. Considering r and z as functions of ϕ, we take the couple of ODEs: $dr/d\phi = rB_r(r,\phi,z)/B_\phi(r,\phi,z)$ and $dz/d\phi = rB_z(r,\phi,z)/B_\phi(r,\phi,z)$; they have been solved numerically making 400 toroidal revolutions with initial conditions $r(0) = r_0$ and $z(0) = z_0$ with (r_0, z_0) the coordinates of several points inside the plasma region.

Construction of 3D equilibrium

Applying the above procedure we have constructed several equilibria which showcase closed and nested toroidal magnetic surfaces. They all have a magnetic axis located, as in the axisymmetric case, at $(r = 1,\ z = 0)$ independently of ϕ; also, they form a separatrix similar in shape with the axisymmetric one, which, depending on the values of the free parameters and ϕ, can be either inside or outside the axisymmetric separatrix. The current-density surfaces, not coinciding with the magnetic surfaces, can be closed and nested too. In addition, the equilibria exhibit closed and nested isomagnetic surfaces. The radial position, r^B_{ax}, of the isomagnetic axis, depending on the free parameters δ, ϵ and F_0 in a similar way as in the axisymmetric case, it also depends on the perturbation parameters c_h and d_h and it can vary with the toroidal coordinate ϕ.
An equilibrium example strongly perturbed via the function $g(\phi)$ is given in Fig. 2 presenting Poincaré surfaces of section for $\mathbf{B}$ on $\phi = 0, \pi/4, \pi/2$ together with the 3D magnetic surfaces. The isomagnetic axis of this equilibrium is located outside the separatrix.

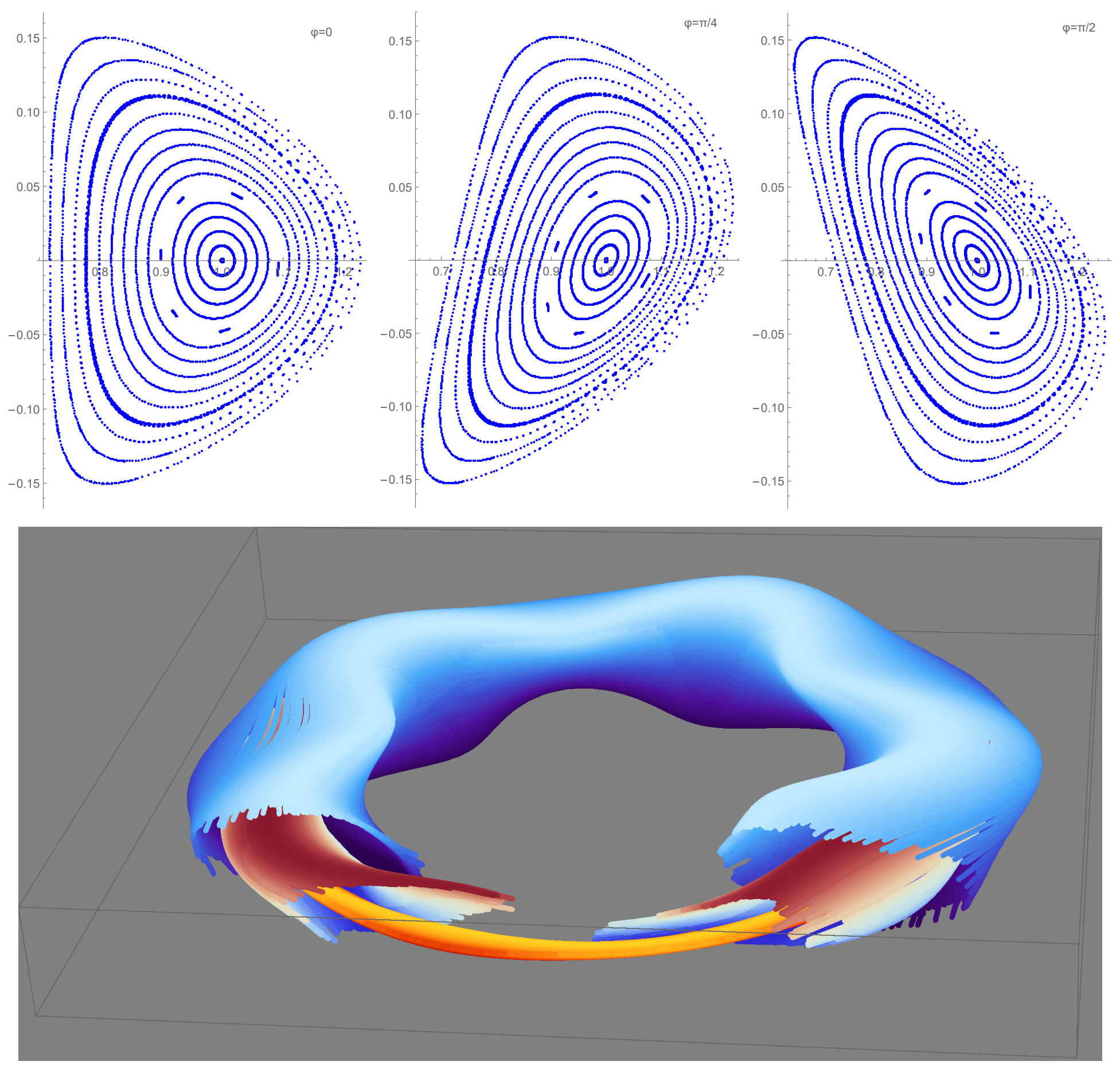

Figure 2: Poincaré plots of a strongly toroidally asymmetric equilibrium on three poloidal cross-sections and the respective 3D depiction of the magnetic surfaces for $\delta = 0.4$, $\epsilon = 0.5$, $F_0 = 3.5$, $c_s = d_s = 0$ $(s = h, w)$, $c_g = 20$, $m_g = 5$ and $d_g = 0$.

For certain parametric values a subregion appears in the outer part of the plasma region close to the separatrix with stochastic magnetic field lines or/and magnetic islands. An example is shown in the upper-left panel of Fig. 3. The respective isomagnetic axis lies inside the separatrix. While in the region close to the separatrix the isomagnetic contours are closed and nested, the magnetic field in the same region is stochastic. Therefore, the existence of closed and nested isomagnetic surfaces within the plasma region is not sufficient for the existence of closed and nested magnetic surfaces.

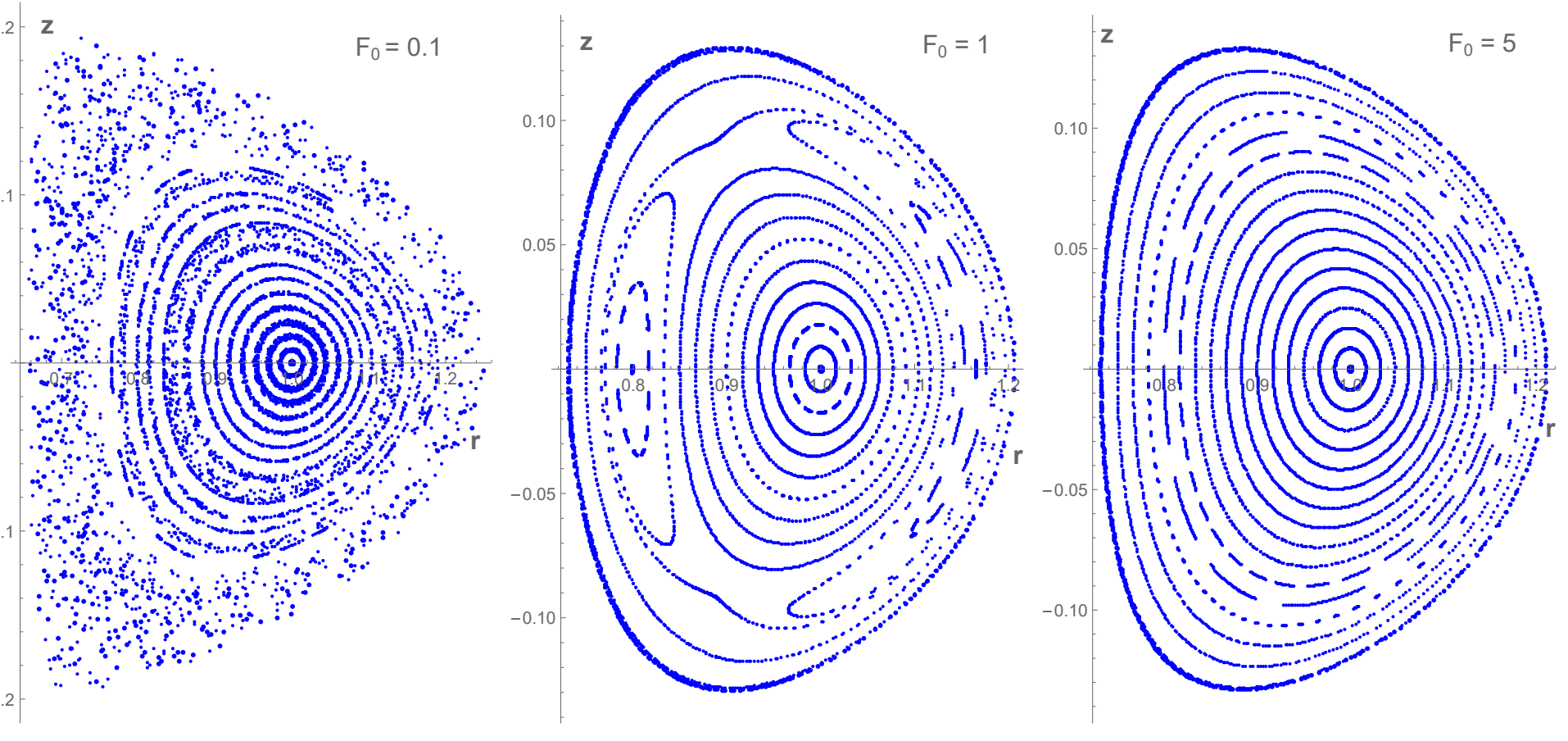

Figure 3: Left panel: Poincaré plot of the magnetic-field lines for $\phi = 0$ of an equilibrium having a large stochastic outer subregion, while in the central region the magnetic surfaces remain closed and nested for $\delta = \epsilon = 0.4$, $F_0 = 0.1$, $c_g = 0.04$, $m_g = 1$, $d_g = 0$ and $h(\phi) = w(\phi) \equiv 0$. Medium and right panels: Respective Poincaré plots for $F_0 = 1$ and $F_0 = 5$. As F_0 increases, the stochastic region shrinks and magnetic islands are formed and eventually disappear.

Conclusions

- We have constructed a special class of 3D toroidal equilibria with anisotropic pressure that showcase closed, nested magnetic surfaces, via harmonic toroidal perturbations of arbitrary amplitude to the axisymmetric Solov'ev equilibrium (Eqs. (4-7)).
- The magnetic surfaces, having a separatrix, depart from the current-density surfaces which are closed and nested too and have a distinct separatrix which coincides with the axisymmetric magnetic one.
- The equilibria exhibit closed and nested isomagnetic surfaces with the isomagnetic axis positioned either inside or outside the magnetic separatrix.
- It has been demonstrated that the existence of closed and nested isomagnetic surfaces inside the plasma region is neither necessary nor sufficient for the existence of respective closed and nested magnetic surfaces.
- For certain values of the free parameters an area of stochastic magnetic-field lines potentially including magnetic islands is formed in the outer region close to the separatrix, while closed and nested magnetic surfaces persist in the central region. The extent of this area depends on the free parameters; in particular, it drastically shrinks as the vacuum toroidal magnetic field assumes larger values.

Acknowledgment: This work was conducted in the framework of participation of the University of Ioannina in the National Programme for the Controlled Thermonuclear Fusion, Hellenic Republic.

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