Three-dimensional steady states as perturbations of the Solov'ev equilibrium

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Motivation and Aim

Magnetic confinement of laboratory fusion plasmas involves the existence of well defined closed and nested toroidal magnetic surfaces. For two-dimensional (2D) magnetohydrodynamic (MHD) equilibria, e.g. in the presence of axisymmetry, the existence of such surfaces is rigorously guaranteed. However, in the absence of any continuous spatial symmetry the existence of steady states with nested toroidal magnetic surfaces has been questioned because the breaking of symmetry allows for magnetic field-line braiding [1]. For this reason, in order to obtain steady states with favourable confinement properties, particularly in the framework of stellator optimization, some kinds of symmetry have been imposed, e.g. quasisymmetry in which B having a continuous symmetry in certain coordinate systems, e.g. Boozer coordinates, becomes independent of one of the coordinates [2]. Recently, weakly toroidally asymmetric plasma equilibria with nested magnetic surfaces and isotropic pressure were constructed by a method that expands up to first order about the axisymmetric configuration without any symmetry assumption. This indicates the existence of analytical counter examples to the aforementioned conjecture of non-existence of such 3D equilibria [3].

The aim of the present study is to construct 3D toroidal equilibria with pressure anisotropy, exhibiting closed and nested magnetic surfaces on the basis of two foundations: first, a special class of equilibria with anisotropic pressure components given by Eqs. (4) valid for any (asymmetric) magnetic field, which was identified for the first time in [4]; and second, by introducing the representation (6) for the magnetic field in cylindrical coordinates (r,ϕ,z).

Axisymmetric Solov'ev equilibrium

The Grad-Shafranov equation describing axisymmetric MHD equilibria is written in the dimensionless form:

$$\Delta^\star\psi + F\frac{dF}{d\psi} + r^2\frac{dP}{d\psi} = 0\,, \quad \Delta^\star := \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r}\frac{\partial}{\partial r}, \tag{1}$$

where (r,ϕ,z) are cylindrical coordinates and $\psi(r,z)$ is the poloidal magnetic flux function which labels the magnetic surfaces. Choosing the free functions as $P(\psi) = P_a - 2(1+\delta^2)\psi$ and $F(\psi) = (F_0^2 + 4\varepsilon\psi)^{1/2}$, where δ, ε, F_0 and P_a are free parameters, the resulting equation admits the Solov'ev solution:

$$\psi_{ax} = z^2(r^2-\varepsilon) + \frac{\delta^2}{4}(r^2-1)^2. \tag{2}$$

The respective up-down symmetric equilibrium is diamagnetic for $\varepsilon \geq 0$ and paramagnetic for $\varepsilon < 0$. In the present study we restrict ourselves to the diamagnetic configuration, shown in Fig. 1-left. It forms spontaneously a separatrix consisting of an outer elliptic part and an inner part parallel to the axis of symmetry, thus having a couple of X-points. The magnetic axis is located at $(z=0,\ r=1)$ on which $\psi_{ax}=0$. For $\varepsilon > 0$ the solution describes a tokamak-type equilibrium, while for $\varepsilon = 0$ reduces to a spheromak one. Since the plasma boundary is not imposed from the

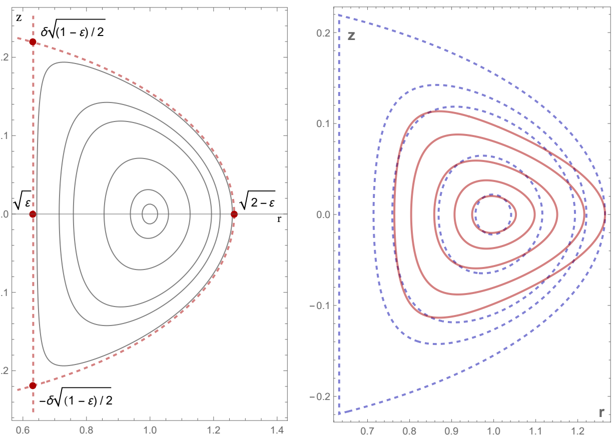

Figure 1: Left panel: The axisymmetric diamagnetic Solov'ev configuration for $\delta = \varepsilon = 0.4$. Right panel: Respective isomagnetic contours (red-continuous curves) with the outer curve touching the separatrix. The blue-dashed curves represent the respective intersections of the magnetic surfaces with the poloidal plane.

outset, it can be chosen a posteriori to coincide with the separatrix or with one of the internal, smoothly closed magnetic surfaces.

The equilibrium has nested surfaces of constant magnetic-field modulus (isomagnetic surfaces) without forming a separatrix with the isomagnetic axis located at ($z = 0$, $r = r_{ax}^B$); the radial coordinate, r_{ax}^B, can be obtained analytically in terms of the free parameters δ, ε and F_0. r_{ax}^B strongly increases with F_0 and weakly decreases with δ and ε. Therefore, pending on the values of the free parameters, the position of the isomagnetic axis can be either inside or outside the magnetic separatrix. An example of the former case is given in Fig. 1-right.

The framework of 3D equilibria

We now consider 3D equilibria with anisotropic pressure governed by the equations:

$$\nabla \cdot \mathscr{P} + \mathbf{B} \times \mathbf{j} = 0, \quad \nabla \times \mathbf{B} = \mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0, \tag{3}$$

where $\mathscr{P} = P_\perp \mathscr{I} + (P_\parallel - P_\perp)/B^2 \mathbf{BB}$ is the pressure tensor consisting of one element representing the parallel to the magnetic field pressure component ($P_\parallel$) and two equal perpendicular components ($P_\perp$) with $\mathscr{I}$ the identity tensor.
In [4], a special class of solutions was identified valid for any magnetic field, namely:

$$P_\perp = P_0 - \frac{1}{2}B^2, \quad P_\parallel = P_0 + \frac{1}{2}B^2. \tag{4}$$

In view of (4), any divergence-free magnetic field satisfies the force-balance equation. However, as stated in [4], this MHD solution does not represent confined plasmas, since P_0 is a constant pressure. In the above context, we introduce the following magnetic-field representation, which is by construction divergence-free:

$$\mathbf{B} = \nabla(\phi + w(\phi)) \times \nabla U(r,\phi,z) + I(r,z)\nabla\phi\,; \tag{5}$$

The free functions in (5) are specified as follows:

$$U(r,\phi,z) = z^2[r^2 - \varepsilon(1+g(\phi))] + \frac{\delta^2(1+h(\phi))}{4}(r^2-1)^2. \tag{6}$$

$$s(\phi) = c_s\cos(m_s\phi) + d_s\sin(n_s\phi) \quad (s = g,h,w), \quad I(r,z) = (F_0^2 + 4\varepsilon\psi_{ax})^{1/2}, \tag{7}$$

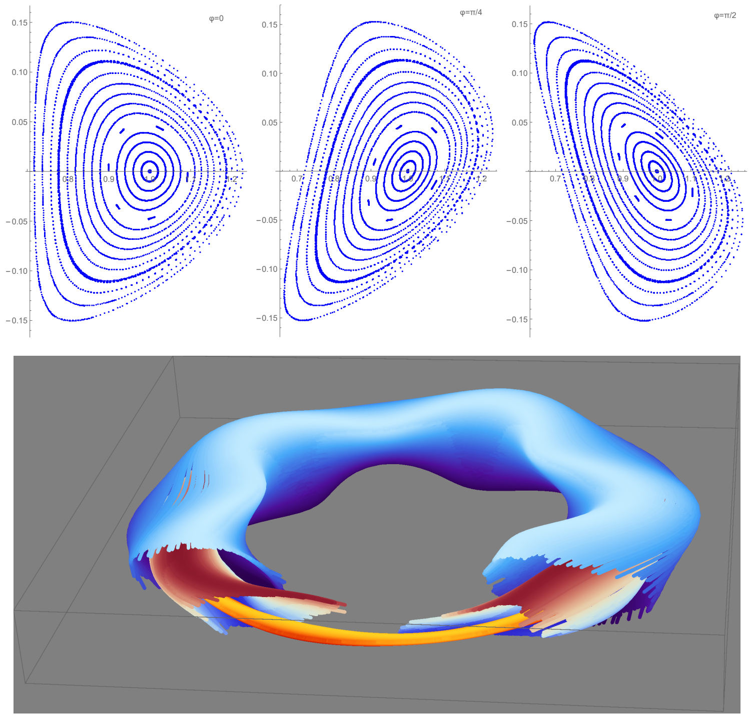

Figure 2: Poincaré surfaces of section of a strongly toroidally asymmetric equilibrium on three poloidal cross-sections and the respective 3D depiction of the magnetic surfaces for $\delta = 0.4$, $\varepsilon = 0.5$, $F_0 = 3.5$, $c_s = d_s = 0$ $(s = h, w)$, $c_g = 20$, $m_g = 5$ and $d_g = 0$.

with ψ_{ax} given by (2).

To examine whether the equilibria determined by Eqs. (4)-(7) can have closed and nested magnetic surfaces we have traced out the magnetic field lines on the basis of the equation $d\mathbf{x}(l)/dl = \mathbf{B}(\mathbf{x})$, where l is the arc-length associated with the vector $\mathbf{x}$ tangential to $\mathbf{B}$. Considering r and z as functions of ϕ, we take the couple of ODEs: $dr/d\phi = rB_r(r,\phi,z)/B_\phi(r,\phi,z)$ and $dz/d\phi = rB_z(r,\phi,z)/B_\phi(r,\phi,z)$; they have been solved numerically making 400 toroidal revolutions with initial conditions $r(0) = r_0$ and $z(0) = z_0$ with (r_0, z_0) the coordinates of several points inside the plasma region.

Construction of 3D equilibria

Applying the above procedure we have constructed several equilibria which showcase closed and nested toroidal magnetic surfaces. They all have a magnetic axis located, as in the axisymmetric case, at $(r = 1,\ z = 0)$ independently of ϕ; also, they form a separatrix similar in shape with the axisymmetric one, which, depending on the values of the free parameters and ϕ, can be either inside or outside the axisymmetric separatrix. The current-density surfaces, not coinciding with the magnetic surfaces, can be closed and nested too. In addition, the equilibria exhibit closed and nested isomagnetic surfaces. The radial position, r^B_{ax}, of the isomagnetic axis depending on the free parameters δ, ε and F_0 in a similar way as in the axisymmetric case, it also depends on the perturbation parameters c_h and d_h and it can vary with the toroidal coordinate ϕ. An equilibrium example strongly perturbed via the function $g(\phi)$ is given in Fig. 2 presenting Poincaré surfaces of section for $\mathbf{B}$ on $\phi = 0, \pi/4, \pi/2$ together with the 3D magnetic surfaces. The isomagnetic axis of this equilibrium is located outside the separatrix.

For certain parametric values a subregion appears in the outer part of the plasma region close

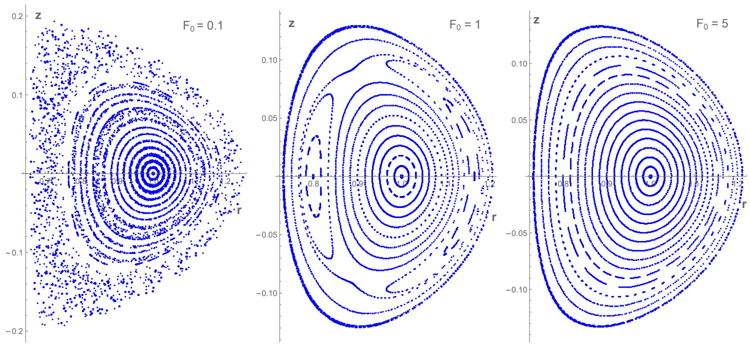

Figure 3: Left panel: Poincaré plot of the magnetic-field lines for $\phi = 0$ of an equilibrium having a large stochastic outer subregion, while in the central region the magnetic surfaces remain closed and nested for $\delta = \varepsilon = 0.4$, $F_0 = 0.1$, $c_g = 0.04$, $m_g = 1$, $d_g = 0$ and $h(\phi) = w(\phi) \equiv 0$. Medium and right panels: Respective Poincaré plots for $F_0 = 1$ and $F_0 = 5$. As F_0 increases the stochastic region shrinks, magnetic islands are formed and eventually disappear.

to the separatrix with stochastic magnetic field lines or/and magnetic islands. An example is shown in the upper-left panel of Fig. 3. The respective isomagnetic axis lies inside the separatrix. While in the region close to the separatrix the isomagnetic contours are closed and nested, the magnetic field in the same region is stochastic. Therefore, the existence of closed and nested isomagnetic surfaces within the plasma region is not sufficient for the existence of closed and nested magnetic surfaces.

Conclusions

We have constructed a special class of 3D toroidal equilibria with anisotropic pressure that showcase closed, nested magnetic surfaces, via harmonic toroidal perturbations of arbitrary amplitude to the axisymmetric Solov'ev equilibrium (Eqs. (4-7)). The magnetic surfaces, having a separatrix, depart from the current-density surfaces which are closed and nested too and have a distinct separatrix which coincides with the axisymmetric magnetic one. The equilibria exhibit closed and nested isomagnetic surfaces with the isomagnetic axis positioned either inside or outside the magnetic separatrix. It has been demonstrated that the existence of closed and nested isomagnetic surfaces inside the plasma region is neither necessary nor sufficient for the existence of respective closed and nested magnetic surfaces. For certain values of the free parameters an area of stochastic magnetic-field lines potentially including magnetic islands is formed in the outer region close to the separatrix, while closed and nested magnetic surfaces persist in the central region. The extent of this area depends on the free parameters; in particular, it drastically shrinks as the vacuum toroidal magnetic field assumes larger values.

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