Stationary Power-Law Solutions of Weak Kinetic-Alfvénic Turbulence

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Abstract

A wave-kinetic description of weak kinetic-Alfvénic turbulence based on gyrokinetics is proposed. The wave kinetic equation for resonant three-wave interactions is derived, and stationary spectra are obtained analytically via Zakharov transformations in both the short- and long-wavelength limits for co- and counter-propagating kinetic Alfvén waves. The cascade directions are identified and verified by numerical solution of the wave kinetic equation. Relevance to solar wind turbulence is briefly discussed.

Introduction

Turbulent cascades of kinetic Alfvén waves (KAWs) are central to understanding the spectra and heating of the solar wind. At kinetic scales, KAWs become dispersive and can transfer energy across the ion gyro-scale through nonlinear interactions among both counter- and co-propagating wave-packets. Observations suggest a correlation between spectral steepening and imbalance, often quantified by the helicity invariant. While counter-propagating cascades are well studied in weak turbulence theory, the role of co-propagating KAW interactions, which conserve both energy and a sign-definite helicity, remains less explored. The presence of two positive-definite invariants opens the possibility of dual cascades, analogous to two-dimensional hydrodynamics. Here we present a systematic wave-kinetic analysis of weak KAW turbulence [1], deriving stationary Kolmogorov-Zakharov (KZ) spectra for both co- and counter-propagating cases and verifying the predicted dual cascade behaviour numerically.

Gyrokinetic Model and Conserved Quantities

In the low-β limit ($m_e/m_i \ll \beta_e \sim \beta_i \ll 1$), the nonlinear dynamics of KAWs is described by the gyrokinetic vorticity equation and the parallel Ohm's law [2]. Using the scalar potential $\delta\phi$ and vector potential $\delta A_\parallel$, and keeping full finite-ion-Larmor-radius (FILR) effects, the system reduces to a single nonlinear mode equation in Fourier space:

$$b_k \varepsilon_{Ak} \delta\phi_k = \int \frac{i\Lambda_{12}^k}{2\omega_k} \beta_{12}^k \delta\phi_1 \delta\phi_2 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2, \quad (1)$$

where $\Lambda_{12}^k = \hat{\mathbf{b}}_0 \cdot \mathbf{k}_{1\perp} \times \mathbf{k}_{2\perp}$, $b_k = k_\perp^2 \rho_i^2$, $\varepsilon_{Ak} = (1-\Gamma_k)/b_k - (k_\parallel^2 v_A^2/\omega_k^2)\sigma_k$ is the linear dielectric operator, $\sigma_k = 1 + \tau(1-\Gamma_0)$ with $\tau = T_e/T_i$, and $\Gamma_k = I_0(b_k)e^{-b_k}$. The nonlinear coupling coefficient is

$$\beta_{12}^k = v_A^2 \sigma_1 \sigma_2 \left(\frac{k_{2\parallel}}{\omega_2} - \frac{k_{1\parallel}}{\omega_1} \right) \left(b_k \frac{k_\parallel}{\omega_k} + b_1 \frac{k_{1\parallel}}{\omega_1} + b_2 \frac{k_{2\parallel}}{\omega_2} \right). \tag{2}$$

Equation (1) conserves two quadratic invariants [3, 4]: the generalized free energy

$$E = \int d\mathbf{r} \left[\frac{n_0 e^2}{T_i} \frac{\sigma_0(1-\Gamma_0)|\delta\phi|^2}{2} + \frac{|\nabla_\perp \delta A_\parallel|^2}{8\pi} \right], \tag{3}$$

and the generalized helicity

$$P = \int d\mathbf{r}\, \delta A_\parallel \frac{1-\Gamma_0}{\rho_i^2} \delta\phi. \tag{4}$$

In the co-propagating regime ($k_\parallel > 0$ for all modes), P is sign-definite, implying two positive quadratic invariants and the possibility of dual cascades.

Wave Kinetic Equation

Introducing the wave-action amplitude $c_k = \sqrt{\sigma_k(1-\Gamma_k)/|\omega_k|}\,\phi_k$ and applying the random-phase approximation yields the wave kinetic equation (WKE) [5]

$$\partial_t n_k = \int d\mathbf{k}_1 d\mathbf{k}_2 \left(\mathscr{R}_{12}^k - \mathscr{R}_{2k}^1 - \mathscr{R}_{k1}^2 \right), \tag{5}$$

with $\mathscr{R}_{12}^k = 4\pi |V_{12}^k|^2 (n_1 n_2 - n_k n_1 - n_k n_2)\delta(\omega_k - \omega_1 - \omega_2)\,\delta_{12}^k$, and V_{12}^k derived from (1), as shown in [1]. The resonance conditions restrict the interaction geometry, as shown in Fig. 1. For co-propagating KAWs, the parallel momentum conservation makes $P = \int k_\parallel n_k d\mathbf{k}$ a positive invariant.

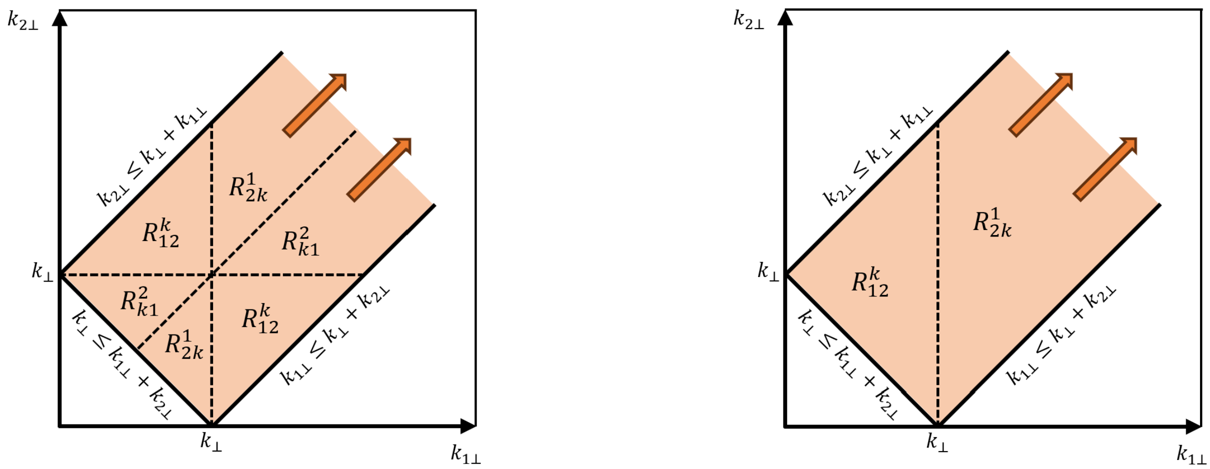

Figure 1: Integration regions in $k_\perp$-space for (a) co-propagating and (b) counter-propagating KAWs. Resonance conditions split the collision integral into distinct branches.

KZ spectra and Cascade Directions

Assuming bi-homogeneous exponents for the real frequency ω_k and the interaction coefficient $|V_{12}^k|$ as $(\alpha_\parallel, \alpha_\perp)$ and $(\beta_\parallel, \beta_\perp)$ respectively, the stationary power-law solutions $n_k \propto k_\parallel^{\nu_\parallel} k_\perp^{\nu_\perp}$ are obtained by Zakharov transformations [5]. Equating the exponents to those of a constant energy flux $(\mu_\parallel = \alpha_\parallel, \mu_\perp = \alpha_\perp)$ and a constant parallel-momentum (helicity) flux $(\mu_\parallel = 1, \mu_\perp = 0)$ gives the KZ spectra

$$\text{energy cascade:} \qquad n_k \propto k_\parallel^{-\beta_\parallel - 1} k_\perp^{-\beta_\perp - 2}, \tag{6}$$

$$\text{momentum cascade:} \qquad n_k \propto k_\parallel^{(\alpha_\parallel - 2\beta_\parallel - 3)/2} k_\perp^{(\alpha_\perp - 2\beta_\perp - 4)/2}. \tag{7}$$

The cascade directions (direct for energy, inverse for momentum) are determined from the sign of the perpendicular fluxes computed via the dimensionless collision integral [1, 5].

Analytical Spectra

In the short-wavelength limit ($k_\perp \rho_i \gg 1$), $\Gamma_k \simeq 0$, $\omega_k \propto k_\parallel k_\perp$ $(\alpha_\parallel = 1, \alpha_\perp = 1)$ and $|V_{12}^k|$ scales as $(\beta_\parallel, \beta_\perp) = (1/2, 5/2)$ for both co-propagating and counter-propagating cases. In the long-wavelength limit ($k_\perp \rho_i \ll 1$, co-prop), $\omega_k - k_\parallel v_A \propto k_\parallel k_\perp^2$ $(\alpha_\parallel = 1, \alpha_\perp = 2)$ and $|V_{12}^k| \propto (\sin\theta_k / k_\perp)|k_{1\perp}^2 - k_{2\perp}^2|(k_\perp^2 + k_{1\perp}^2 + k_{2\perp}^2)$ with $(\beta_\parallel, \beta_\perp) = (1/2, 3)$. For the counter-propagating case, the usual reduced-MHD results give $\omega_k \propto k_\parallel$, and $|V_{12}^k| \propto (\sin\theta_k / k_\perp)(k_{1\parallel} k_{2\parallel} / k_\parallel)^{1/2} |k_\perp^2 + k_{1\perp}^2 - k_{2\perp}^2|$ yielding $(\beta_\parallel, \beta_\perp) = (1/2, 1)$. The resulting KZ exponents are summarized in Table 1.

Table 1: Scaling exponents and KZ spectra for weak KAW turbulence.

Regime	Invariant	$(\nu_\parallel, \nu_\perp)$	Direction
$k_\perp \rho_i \gg 1$ (co)	energy	$(-3/2, -9/2)$	direct
	momentum	$(-3/2, -4)$	inverse
$k_\perp \rho_i \ll 1$ (co)	energy	$(-3/2, -5)$	direct
	momentum	$(-3/2, -4)$	inverse
$k_\perp \rho_i \ll 1$ (counter)	energy	$(-3/2, -3)$	direct

Numerical Verification

The WKE is solved for the co-propagating case assuming $n_k \propto k_\parallel^{-3/2} n(k_\perp)$. Direct energy-cascade spectra in the long- and short-wavelength limits ($k_\perp^{-5}$ and $k_\perp^{-9/2}$) are well verified when a large-scale Gaussian injection is applied (not shown) [1]. To demonstrate the dual cascade behavior, a small-scale Gaussian injection centered at $k_\perp \rho_i = 10$ is used with FILR effects fully retained. Figure 2 shows the temporal evolution of $n(k_\perp)$. A dual cascade develops: towards large scales the spectrum steepens to $\sim k_\perp^{-5}$ (inverse momentum-cascade) and towards small scales it settles to $\sim k_\perp^{-9/2}$ (direct energy-cascade), however with a rather limited span. The spectral index in the intermediate range varies between -4 and -5, within analytical predictions. Further numerical investigations will be conducted to deliver a detailed comparison with theoretical results.

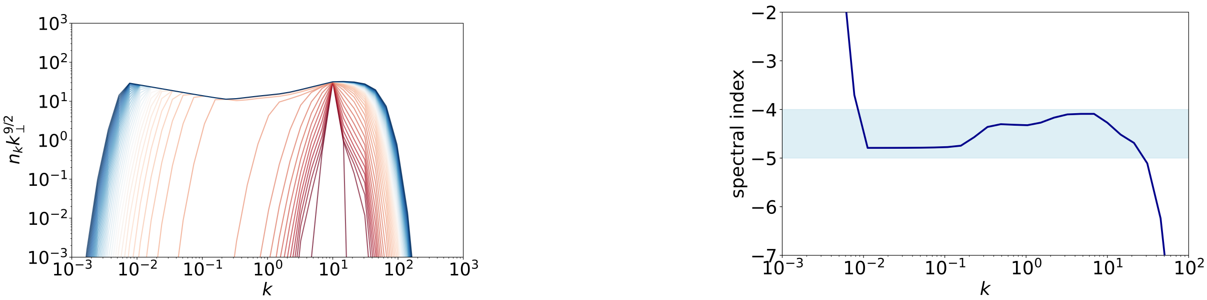

Figure 2: Left: Time evolution of $n(k_\perp)$ with a small-scale injection, showing a dual cascade. Right: Corresponding spectral index; the saturated index varies from ≈ -4 at large scales to $\approx -9/2$ at small scales.

Conclusions

We have presented a weak-turbulence theory for KAWs retaining full FILR effects. Stationary KZ spectra are derived analytically, and direct energy cascades together with an inverse helicity (parallel-momentum) cascade are found for co-propagating KAWs. The inverse cascade as a consequence of the second positive invariant is preliminarily verified numerically. In the long-wavelength (co-propagating) case, the energy spectrum $E(k_\perp) \propto k_\perp^{-4}$ offers a possible explanation for the steep transition range observed in the solar wind. Future work will address nonlocal interactions and the transition between weak and strong turbulence.

References

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