Stationary power-law solutions of weak kinetic-Alfvénic turbulence

BSAP04

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arXiv:2508.03478

ABSTRACT & MOTIVATION

- Alfvénic turbulence is ubiquitous in space/astro/lab plasmas (solar wind, corona, fusion devices)
- In low-β plasma at ion-kinetic scales, kinetic Alfvén waves (KAWs) dominate $\Longrightarrow$ co-propagating KAWs can interact nonlinearly due to finite-ion-Larmor-radius (FILR) effect [Shen 2024 PoP]
- Two invariants (generalized free-energy E and helicity P) suggest possible dual cascade$\Longrightarrow$ inverse cascade due to helicity
- **Goal**: derive wave kinetic equation, find stationary power-law spectra, identify cascade directions, verify numerically

THEORETICAL FRAMEWORK

Model Equations

- Gyrokinetic formulation for KAWs ($\omega_k \ll \Omega_{ci}, \frac{m_e}{m_i} \ll \beta_e \sim \beta_i \ll 1$); perturbed fields $\delta\phi, \delta A_\parallel$
- Generalized parallel Ohm's law & gyrokinetic vorticity equation

$$c^{-1}\partial_t \delta A_\parallel + \nabla_\parallel \sigma_0 \delta\phi = B_0^{-1}[\delta A_\parallel, \sigma_0\delta\phi]$$

$$(\Gamma_0 - 1)\partial_t\delta\phi + \frac{\rho_i^2 v_A^2}{c}\nabla_\parallel \Delta_\perp \delta A_\parallel = -\frac{c}{B_0}[\delta\phi, \Gamma_0\delta\phi] + \frac{\rho_i^2 v_A^2}{cB_0}[\delta A_\parallel, \Delta_\perp \delta A_\parallel]$$

- Nonlinear mode equation (Fourier form) assuming normal-mode

$$b_k \epsilon_{Ak}\delta\phi_k = \int \frac{i\Lambda_{12}^k}{2\omega_k}\beta_{12}^k \delta\phi_1\delta\phi_2\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

- Two conserved quadratic invariants:

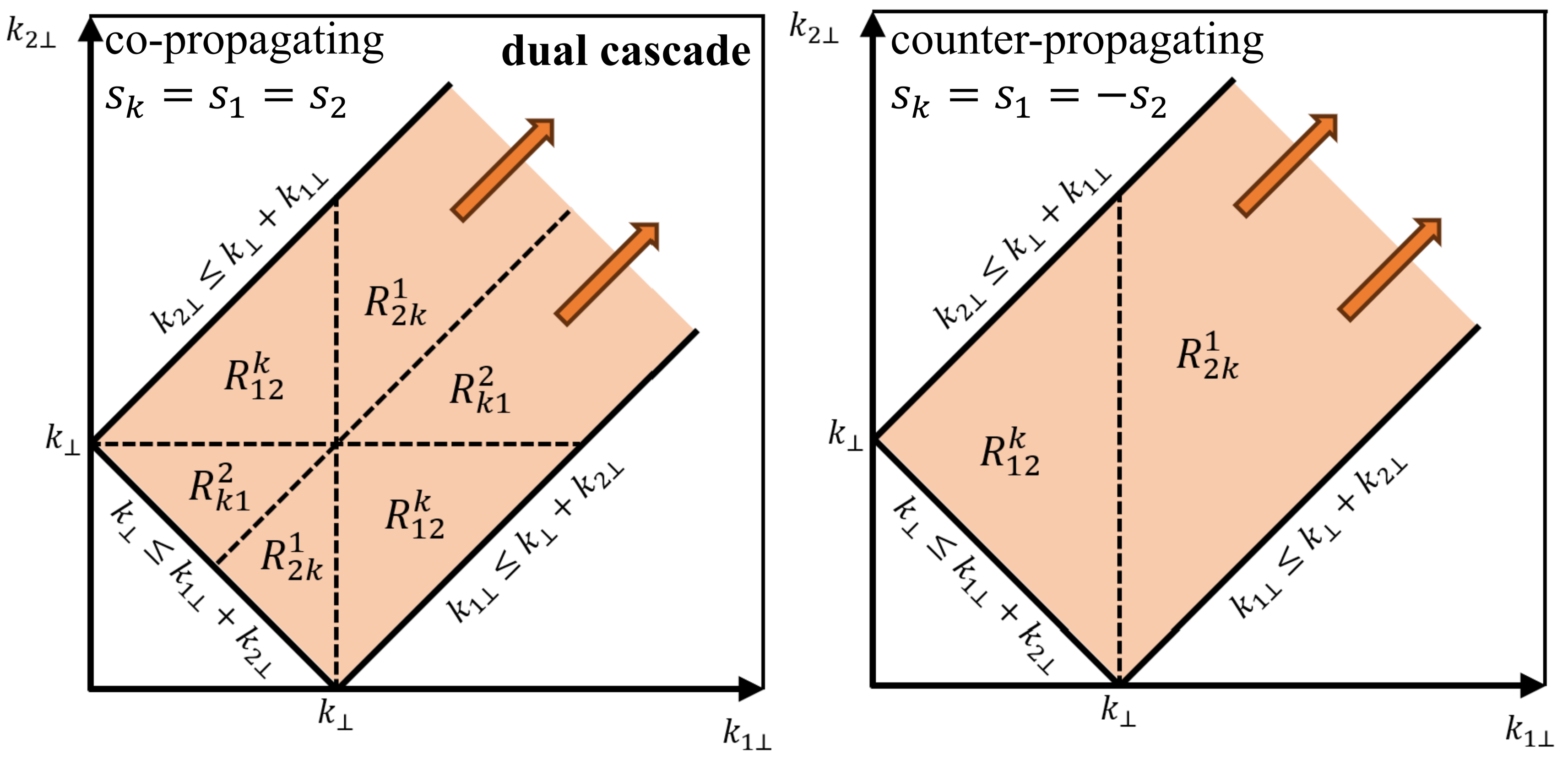
$$E_k = \sigma_k(1-\Gamma_k)|\delta\phi_k|^2 = |\omega_k| n_k, \qquad P_k = k_\parallel n_k \quad \Longrightarrow n_k = E_k/|\omega_k|$$

Method – Weak Turbulence & KZ Spectra

- Wave kinetic equation (WKE) in canonical form [Nazarenko 2011]

$$\partial_t n_k = \mathcal{C}[n] \equiv \int (\mathcal{R}_{12}^k - \mathcal{R}_{2k}^1 - \mathcal{R}_{k1}^2) d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

$$\mathcal{R}_{12}^k = 4\pi|V_{12}^k|^2(n_1n_2 - n_kn_1 - n_kn_2)\delta(\omega_k - \omega_1 - \omega_2)\delta(s_k k_\parallel - s_1 k_{1\parallel} - s_2 k_{2\parallel})\delta(\boldsymbol{k}_\perp - \boldsymbol{k}_{1\perp} - \boldsymbol{k}_{2\perp})$$

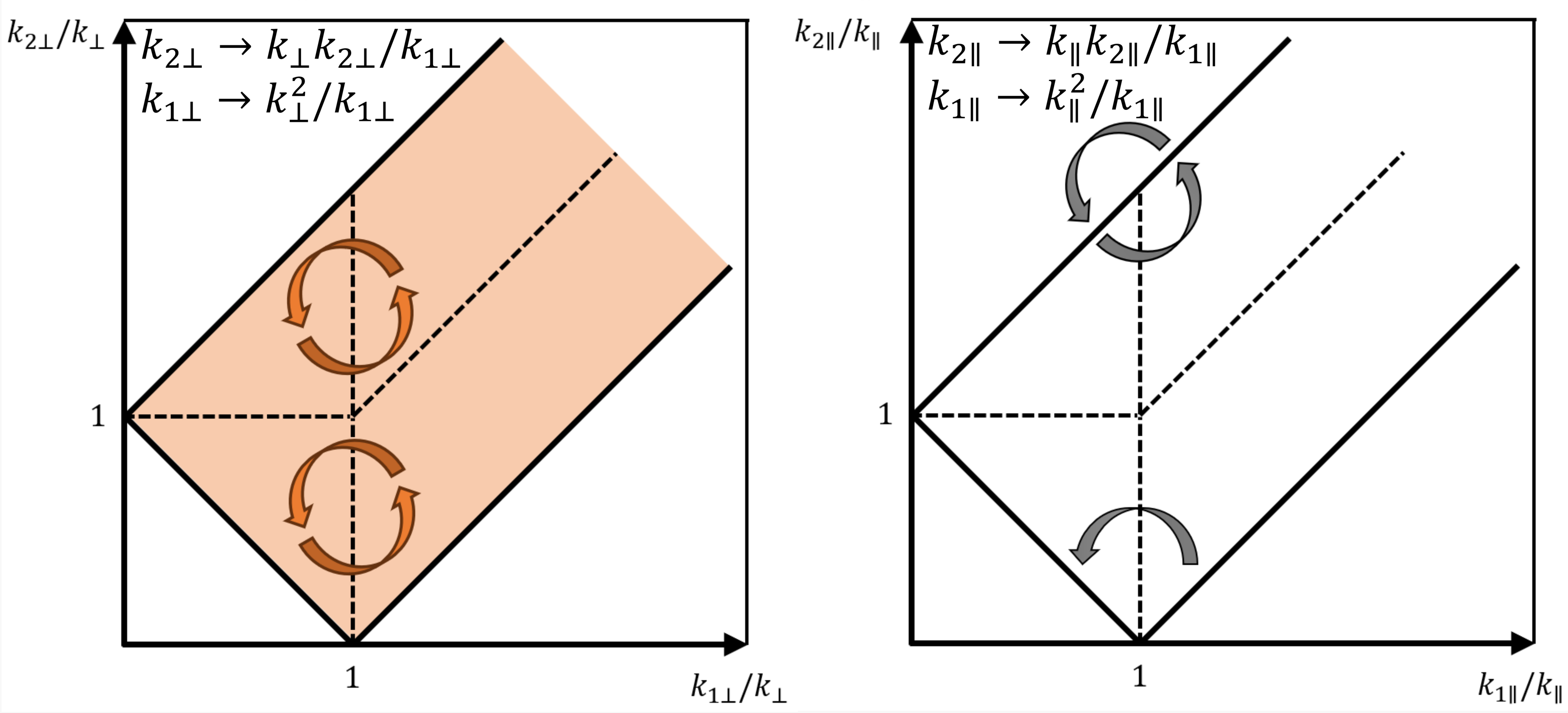


- Zakharov transformation assuming scale-invariance

$\omega_k \propto k_\parallel^{\alpha_\parallel} k_\perp^{\alpha_\perp}, V \propto k_\parallel^{\beta_\parallel} k_\perp^{\beta_\perp}, n_k \propto k_\parallel^{\nu_\parallel} k_\perp^{\nu_\perp}$ (bi-homogeneous functions)

- Kolmogorov-Zakharov spectra (energy & parallel-momentum)

$$n_k \propto k_\parallel^{-\beta_\parallel - 1} k_\perp^{-\beta_\perp - 2}, \qquad n_k \propto k_\parallel^{(\alpha_\parallel - 2\beta_\parallel - 3)/2} k_\perp^{(\alpha_\perp - 2\beta_\perp - 4)/2}$$



KEY RESULTS

Stationary Spectra & Cascade Directions

regime	$\omega_k \propto k_\parallel^{\alpha_\parallel} k_\perp^{\alpha_\perp}$	$V \propto k_\parallel^{\beta_\parallel} k_\perp^{\beta_\perp}$	invariant	$n_k \propto k_\parallel^{\nu_\parallel} k_\perp^{\nu_\perp}$	direction
$k_\perp\rho_i \gg 1$	(1,1)	(1/2,5/2)	energy	(-3/2,-9/2)	direct
$k_\perp\rho_i \gg 1$ (co-)			helicity	(-3/2,-4)	inverse
$k_\perp\rho_i \ll 1$ (co-)	(1,2)	(1/2,3)	energy	(-3/2,-5)	direct
			helicity	(-3/2,-4)	inverse
$k_\perp\rho_i \ll 1$ (counter-)	(1,0)	(1/2,1)	energy	(-3/2,-3)	direct

- $k_\perp\rho_i \ll 1$ (co-prop): nearly scale-invariant [Zakharov 1992]

$$\omega_k - k_\parallel v_A \propto k_\parallel k_\perp^2$$

- Cascade in $k_\perp$ direction $\Longrightarrow$ energy & helicity fluxes [Zakharov 1992]

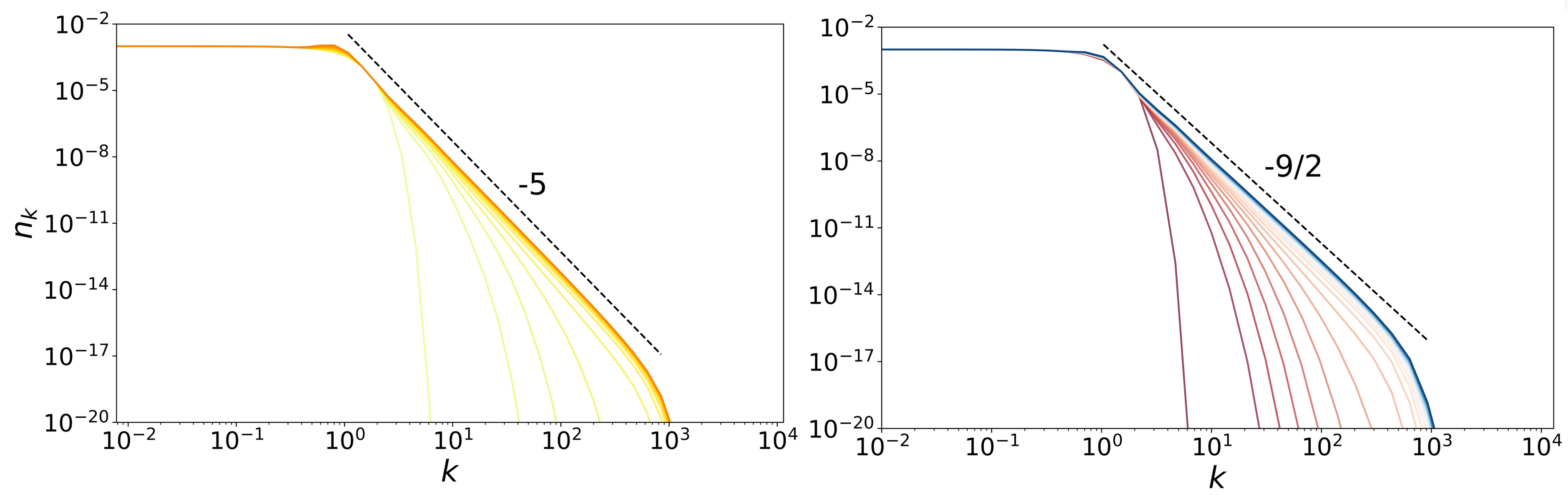
$$\Pi_\perp = -\frac{C_1}{k_\parallel k_\perp}\frac{\partial I(\nu_\parallel,\nu_\perp)}{\partial \nu_\perp}\Bigg|_{\substack{\nu_\parallel=-\beta_\parallel-1\\ \nu_\perp=-\beta_\perp-2}}, R_\perp = -\frac{C_2}{k_\parallel k_\perp}\frac{\partial I(\nu_\parallel,\nu_\perp)}{\partial \nu_\perp}\Bigg|_{\substack{\nu_\parallel=(\alpha_\parallel-2\beta_\parallel-3)/2\\ \nu_\perp=(\alpha_\perp-2\beta_\perp-4)/2}}$$

$I(\nu_\parallel,\nu_\perp)$ is the dimensionless form of $\mathcal{C}[n]$

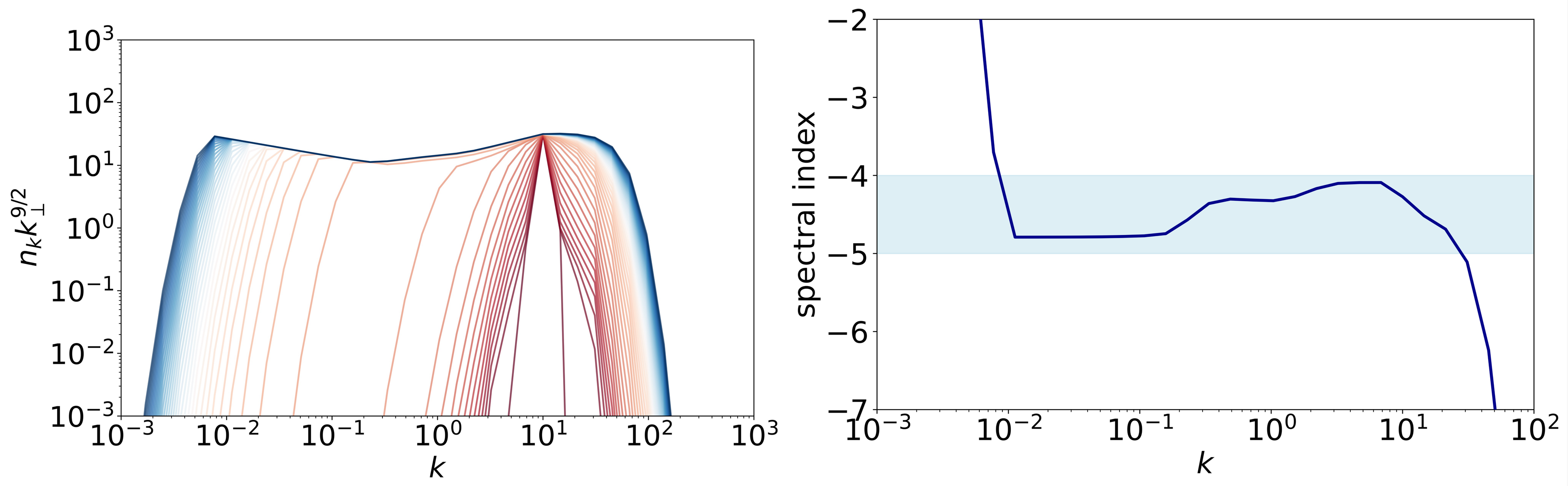
- Co-propagating KAWs exhibit both a direct energy cascade and an inverse helicity (momentum) cascade
- Counter-propagating (balanced) case recovers RMHD/ERMHD results [Galtier 2000, 2015 JPP]
- Potential explanation for steep solar-wind transition-range spectra

Numerical Verification

- Integrate WKE in $k_\perp$ space assuming $n_k \propto k_\parallel^{-3/2} n(k_\perp)$
- Direct energy-cascade spectra (co-prop): $n(k_\perp) \propto k_\perp^{-5}, \ k_\perp^{-9/2}$



- Inverse transfer preliminarily identified



CONCLUSIONS

- Derived WKE for weak KAW turbulence from gyrokinetics
- Found stationary KZ spectra: direct energy cascade in both co- and counter-propagating cases; inverse helicity cascade only for co-propagating case
- KZ energy spectra for co-propagating case numerically verified
- Numerical solutions show inverse transfer
- Future works: improved numerics; locality of interaction; ST

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