# Determining distribution functions for one-dimensional force-free current sheet equilibria

S. Boswell[1], T. Neukirch[1]

[1] *School of Mathematics and Statistics, University of St Andrews, St Andrews, UK*

Current sheets are ubiquitous in solar, space and astrophysical plasmas (e.g. [1, 2]), which are often collisionless. Observations indicate that in some cases the current sheets seem to be force-free (e.g. [1]). This raises the question how one can determine distribution functions that are self-consistently giving rise to a given force-free magnetic field (and hence also current density) profile.

We will describe current sheets as spatially one-dimensional structures in Cartesian coordinates, with $x$ and $y$ being the two invariant coordinates and all quantities only depending on the spatial coordinate $z$. The magnetic fields considered here have the form

$$\mathbf{B}(z) = B_x(z)\mathbf{e}_x + B_y(z)\mathbf{e}_y, \tag{1}$$

i.e. $B_z = 0$. The force-free condition $\mathbf{j}\times\mathbf{B} = \mathbf{0}$ implies that

$$B_x^2(z) + B_y^2(z) = B_T^2 = \text{constant}. \tag{2}$$

To ensure that the collisionless equilibrium distribution functions $F_s$ for particle species $s$ satisfy the Vlasov equation they should depend on the constants of motion. We shall assume that the distribution functions depend on the three obvious constants of motion associated with the three symmetries of the system (time, $x$-direction, and $y$-direction), namely the Hamiltonian $H_s = m_s v^2/2 + q_s\Phi(z)$, and the two canonical momenta $p_{x,s} = m_s v_x + q_s A_x(z)$ and $p_{y,s} = m_s v_y + q_s A_y(z)$. Here $m_s$ and $q_s$ are the mass and electric charge of a particle of species $s$, $\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z$ is the velocity, $\Phi(z)$ is the electrostatic potential, and $\mathbf{A}(z) = A_x(z)\mathbf{e}_x + A_y(z)\mathbf{e}_y$ is the magnetic vector potential. We remark that we will in the following assume that the system parameters are adjusted in such a way that $\Phi(z) = 0$ throughout (for the problems considered here this can always be achieved).

Channell [3] has developed a method to determine distribution functions by using a component of the pressure tensor which in our chosen coordinate system is the $zz$-component, $P_{zz}$, defined by

$$P_{zz}(A_x, A_y) = \sum_s m_s \iiint v_z^2 F_s(H_s, p_{x,s}, p_{y,s})\, \mathrm{d}^3 v. \tag{3}$$

Assuming that the distribution function has the form

$$F_s = F_{s0}\exp(-\beta_s H_s)\, g_s(p_{x,s}, p_{y,s}), \tag{4}$$

with $\beta_s = (k_B T_s)^{-1}$ where $k_B$ is the Boltzmann constant and $T_s$ a temperature, Eq. (3) takes the form of an integral equation for $g_s$:

$$P_{zz}(A_x, A_y) = C_s \iint \exp\left\{-\frac{\beta_s}{2m_s}\left[(p_{x,s} - q_s A_x)^2 + (p_{y,s} - q_s A_y)^2\right]\right\} g_s(p_{x,s}, p_{y,s}) \mathrm{d}p_{x,s} \mathrm{d}p_{y,s}, \quad (5)$$

with $C_s$ a constant factor.

This method has been used by Harrison and Neukirch [4] to find the first known distribution function for the force-free version of the Harris sheet [7] defined as

$$B_x = B_\infty \tanh(z/L), \quad B_y = B_\infty \frac{1}{\cosh(z/L)}. \quad (6)$$

(see Fig. 1). We point out that in this case the $B_y$-component (guide field) goes to zero as $z$ goes to infinity.

In order to use Eq. (5) to derive $g_s$ Harrison and Neukirch [4] needed to first find an appropriate form for $P_{zz}(A_x, A_y)$ for this magnetic field. They made use of the fact that for a force-free magnetic field $P_{zz}$ has to be constant as a function of $z$, and that $A_x(z)$ and $A_y(z)$ are invertible. With the ansatz $P_{zz}(A_x, A_y) = P_1(A_x) + P_2(A_y)$ (Fig. 2, left panel) and $g_s(p_{x,s}, p_{y,s}) = g_{s,x}(p_{x,s}) + g_{s,y}(p_{y,s})$ they obtained a separable problem and could determine the distribution functions (Fig. 2, right panel; see [6] for a review and further developments).

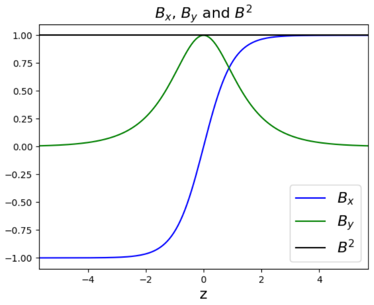

Figure 1: *The magnetic field components of the force-free Harris sheet as functions of* $z/L$ *for the case* $B_y \to 0$ *as* $|z| \to \infty$.

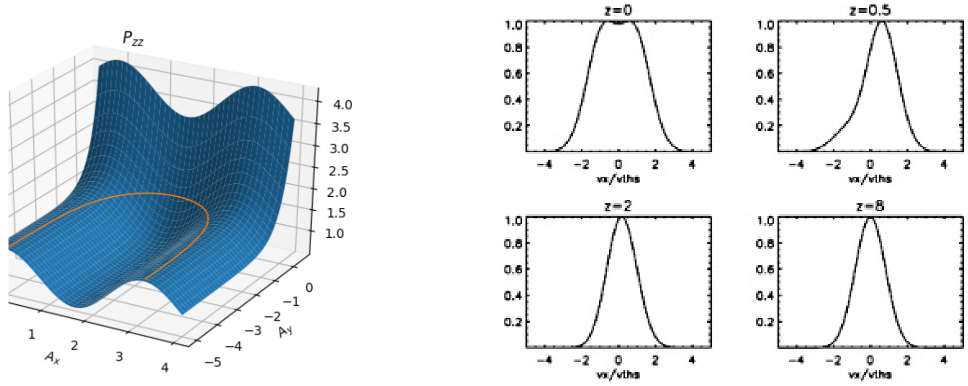

Figure 2: *Left:* $P_{zz}(A_x, A_y)$ *as a surface plot with the vector potential for force-free Harris sheet solution (6) as a isocontour (*$P_{zz}$ *=constant) overplotted. Right: Typical distributions functions (*$v_x$*-direction) for different* $z/L$ *(from [5]).*

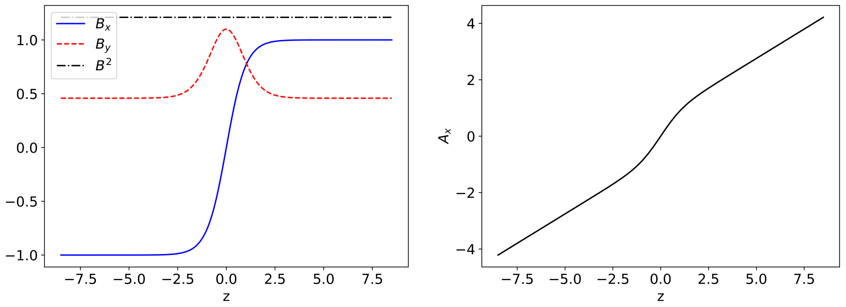

Figure 3: *Left: The magnetic field components of the force-free Harris sheet for the case* $B_y \to$ *constant* $> 0$ *as* $|z| \to \infty$. *Right: The x-component of the vector potential* $A_x(z)$. *Here* $r = B_T/B_\infty = 1.1$ *has been used.*

A relatively simple modification of the magnetic field, however, results in a much more difficult problem. Let us assume that we want $B_y$ to go to a finite value as $|z| \to \infty$, while keeping $B_x(z)$ the same. Then

$$B_y(z) = \pm\sqrt{B_T^2 - B_\infty^2 \tanh^2(z/L)}, \tag{7}$$

with $B_T^2 > B_\infty^2$. Without loss of generality we will in the following only consider the positive root case (see Fig. 3, left panel). By integrating Eq. (7) with respect to $z$ one gets

$$A_x(z) = B_\infty L\left\{\sqrt{r^2-1}\,\mathrm{artanh}\left[\sqrt{r^2-1}\frac{\tanh(z/L)}{\sqrt{r^2-\tanh^2(z/L)}}\right] + \arcsin\left[\frac{\tanh(z/L)}{r}\right]\right\}, \tag{8}$$

where $r = B_T/B_\infty > 1$ (see Fig. 3, right panel). Despite $A_x$ being a one-to-one function of $z$, it is impossible to derive a closed form for the inverse function $z(A_x)$. This implies that contrary to the case when $B_y \to 0$ as $|z| \to \infty$ one cannot find $P_1(A_x)$ in closed form, which obviously presents a problem when wanting to solve the integral equation to derive the particle distribution function. The pressure function is, however, known implicitly (see Fig. 4, left panel)

To overcome this problem we have used the following approach. The properties of $P_1(A_x)$ together with the fact that the range of $A_x$ extends from $-\infty$ to $\infty$ (this is a major difference to the $B_y \to 0$ case) suggests that $g_s(p_{x,s})$ can be expanded into an infinite series of Hermite functions in the form

$$g_{s,x}(p_{x,s}) = g_{s,x,00} + \sum_{n=0}^{\infty} g_{s,x,n}\psi_n(\kappa_s p_{x,s}). \tag{9}$$

Here, $g_{s,x,00}$ and $g_{s,x,n}$ are expansion coefficients that have to be determined and $\psi_n(x)$ are the usual normalised Hermite functions, which are products of a Gaussian function and the Hermite polynomial of order $n$. The reason for this choice is that there is a well-established connection between the action of the Gaussian kernel of the integral equation on Hermite polynomials and the Maclaurin coefficients of $P_1(A_x)$ (see e.g. [8, 9]). One can determine the Maclaurin

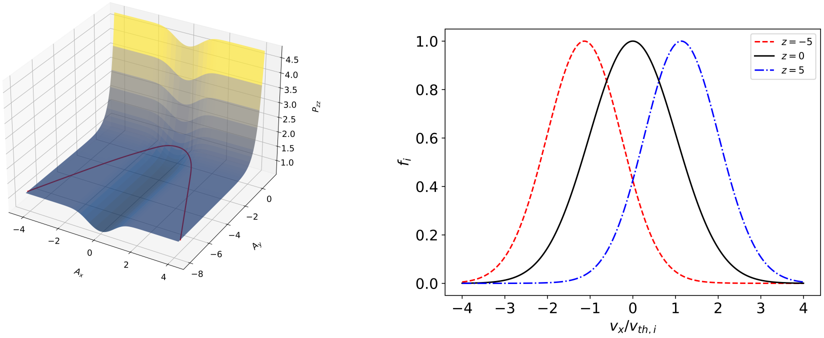

Figure 4: *Left:* $P_{zz}(A_x, A_y)$ *as an implicit surface plot for the case with a nonzero limit of* $B_y$ *as* $|z| \to \infty$. *Right: Examples of distribution functions derived by using an expansion in terms of Hermite functions.*

coefficients of $P_1(A_x)$ by implicit differentiation, for example using Faà di Bruno's formula. The result of a successful calculation, truncating the summation after 30 terms, for the case $r = 1.1$ is shown for three different positions $z/L$ in the right panel of Fig. 4. We remark that a number of different variations of this approach are possible and investigations about advantages and disadvantages of these variations are currently underway.

SB acknowledges support by STFC Doctoral Training Partnership Grant ST/X508779/1. TN acknowledges support by STFC via Consolidated Grant ST/W001195/1.

## References

[1] A V. Artemyev, V. Angelopoulos and I.-Y. Vasko, J. Geophys. Res. A **124**, 2858 (2019)

[2] I.-Y. Vasko, K. Alimov, T. Phan, S. D. Bale, F. S. Mozer and A V. Artemyev, Astrophys. J. Letters **926**, L19 (2022)

[3] P. J. Channell, Phys. Fluids **19**, 1541 (1976)

[4] M. G. Harrison and T. Neukirch, Phys. Rev. Lett. **102**, 135003 (2009)

[5] T. Neukirch, F. Wilson and M. G. Harrison, Phys. Plasmas **16**, 122102 (2009)

[6] T. Neukirch, F. Wilson and O. Allanson, Plasma Phys. Control. Fusion **60**, 014008 (2018)

[7] E. Harris, Nuovo Cimento **23**, 115 (1962)

[8] O. Allanson, T. Neukirch, F. Wilson and S. Troscheit, Phys. Plasmas **22**, 102116 (2015)

[9] O. Allanson, T. Neukirch, S. Troscheit and F. Wilson, J. Plasma Phys. **82**, 905820306 (2016)