Rankine-Hugoniot conditions in wave-aligned Q-variables for MHD discontinuities

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Magnetohydrodynamic (MHD) discontinuities, such as shocks, contact surfaces, tangential discontinuities and rotational discontinuities, are fundamental structures in space and astrophysical plasmas. Across such interfaces, the upstream and downstream plasma states are connected by Rankine-Hugoniot jump conditions, which express conservation of mass, momentum, magnetic flux and energy. In the classical formulation, these conditions are written in the primitive MHD variables $(\rho, \boldsymbol{V}, p, \boldsymbol{B})$. This representation is complete, but it does not explicitly express how the wave content of the plasma responds to the jump.

The Q-variable formalism was introduced as a wave-aligned generalisation of the Elsässer variables. The variables are defined by

$$\boldsymbol{Q}^{\pm} = \boldsymbol{V} \pm \alpha \boldsymbol{B}, \tag{1}$$

where α is a wave-branch parameter. For the Alfvénic choice $\alpha = 1/\sqrt{\mu\rho}$, Eq. (1) reduces to the usual Elsässer representation. More generally, α can be chosen from the characteristic speed of the relevant MHD wave branch. The inverse transformation is

$$\boldsymbol{V} = \frac{\boldsymbol{Q}^{+} + \boldsymbol{Q}^{-}}{2}, \qquad \boldsymbol{B} = \frac{\boldsymbol{Q}^{+} - \boldsymbol{Q}^{-}}{2\alpha}. \tag{2}$$

The purpose of this work to derive the jump conditions directly in the Q-variable formalism. We apply the shock-frame limit to the Q-MHD equations and obtain the full set of Rankine-Hugoniot relations in terms of $\boldsymbol{Q}^{+}$ and $\boldsymbol{Q}^{-}$. We then verify that the resulting system recovers the classical ideal-MHD Rankine-Hugoniot conditions exactly. This establishes that the Q-formalism preserves the full conservation structure of ideal MHD while providing a wave-aligned view of discontinuities.

We consider a planar discontinuity with normal vector $\boldsymbol{n}$ moving with speed u. The jump of a quantity f is denoted by

$$[\![f]\!] = f_1 - f_2, \tag{3}$$

where states 1 and 2 are the upstream and downstream states. In the shock frame, the Q-variables are

$$\boldsymbol{Q}'^{\pm} = \boldsymbol{Q}^{\pm} - u\boldsymbol{n}, \qquad Q_n'^{\pm} = \boldsymbol{Q}'^{\pm} \cdot \boldsymbol{n}, \qquad \boldsymbol{Q}_t'^{\pm} = \boldsymbol{Q}'^{\pm} - Q_n'^{\pm}\boldsymbol{n}. \tag{4}$$

For the wave-frame derivative

$$\frac{D^{\pm} f}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{Q}^{\pm} \cdot \nabla f, \tag{5}$$

integration across the infinitesimal discontinuity gives the shock-limit rule

$$\frac{D^{\pm} f}{Dt} \quad \longrightarrow \quad Q_n'^{\pm} [\![f]\!]. \tag{6}$$

Together with

$$\frac{\partial f}{\partial t} \to -u[\![f]\!], \qquad \nabla f \to \boldsymbol{n}[\![f]\!], \qquad \nabla \cdot \boldsymbol{F} \to \boldsymbol{n} \cdot [\![\boldsymbol{F}]\!], \tag{7}$$

Eq. (6) provides the set of substitutions used to derive the Rankine-Hugoniot conditions from the Q-MHD system.

Applying these shock-limit identities to the Q-MHD equations gives the following jump conditions. The normal magnetic-flux and mass-flux conditions become

$$\left[\!\!\left[\frac{Q_n'^{+} - Q_n'^{-}}{\alpha} \right]\!\!\right] = 0, \tag{8}$$

$$[\![\rho \left(Q_n'^{+} + Q_n'^{-} \right)]\!] = 0. \tag{9}$$

The tangential magnetic-flux condition is

$$\left[\!\!\left[\frac{Q_n'^{-} \boldsymbol{Q}_t'^{+} - Q_n'^{+} \boldsymbol{Q}_t'^{-}}{\alpha} \right]\!\!\right] = 0. \tag{10}$$

The normal momentum condition becomes

$$\left[\!\!\left[\left(Q_n'^{+} + Q_n'^{-} \right)^2 + \frac{4p}{\rho} + \frac{1}{2} \left(1 - \frac{\Delta\alpha^2}{\alpha^2} \right) \left(\boldsymbol{Q}_t'^{+} - \boldsymbol{Q}_t'^{-} \right)^2 \right]\!\!\right] = 0, \tag{11}$$

while the tangential momentum condition is

$$\left[\!\!\left[2Q_n'^{-} \boldsymbol{Q}_t'^{+} + 2Q_n'^{+} \boldsymbol{Q}_t'^{-} + \frac{\Delta\alpha^2}{\alpha^2} \left(Q_n^{+} - Q_n^{-} \right) \left(\boldsymbol{Q}_t'^{+} - \boldsymbol{Q}_t'^{-} \right) \right]\!\!\right] = 0. \tag{12}$$

Here

$$\Delta\alpha^2 = \alpha^2 - \frac{1}{\mu\rho}. \tag{13}$$

Finally, the energy condition is

$$\begin{aligned} \Bigg[\!\!\Bigg[\frac{1}{2} \left(Q_n'^{+} + Q_n'^{-} \right) \left[\left(Q_n'^{+} + Q_n'^{-} \right)^2 + \left(\boldsymbol{Q}_t'^{+} + \boldsymbol{Q}_t'^{-} \right)^2 + \frac{8\gamma}{\gamma - 1} \frac{p}{\rho} \right] \\ + 2 \left(1 - \frac{\Delta\alpha^2}{\alpha^2} \right) \left(Q_n'^{-} \boldsymbol{Q}_t'^{+} - Q_n'^{+} \boldsymbol{Q}_t'^{-} \right) \cdot \left(\boldsymbol{Q}_t'^{+} - \boldsymbol{Q}_t'^{-} \right) \Bigg]\!\!\Bigg] = 0. \end{aligned} \tag{14}$$

Equations (8)–(14) are the Rankine-Hugoniot conditions derived in the Q-variable formalism. They express, respectively, normal magnetic flux, mass flux, tangential magnetic flux, normal momentum, tangential momentum and energy conservation.

To verify consistency, we substitute Eq. (2) into the Q-variable jump conditions. The first two conditions reduce immediately to

$$[\![B_n]\!] = 0, \qquad [\![\rho V_n']\!] = 0, \tag{15}$$

where $V_n' = V_n - u$. The tangential magnetic-flux condition becomes

$$[\![V_n' \boldsymbol{B}_t - B_n \boldsymbol{V}_t]\!] = 0. \tag{16}$$

The momentum equations recover

$$\left[\!\!\left[\rho V_n'^2 + p + \frac{B_t^2}{2\mu}\right]\!\!\right] = 0, \qquad \left[\!\!\left[\rho V_n' \boldsymbol{V}_t - \frac{B_n \boldsymbol{B}_t}{\mu}\right]\!\!\right] = 0, \tag{17}$$

and the energy relation reduces to the usual ideal-MHD energy-flux condition. Thus, the Q-variable jump system is exactly equivalent to the classical Rankine-Hugoniot system, but reorganised in terms of wave-aligned variables.

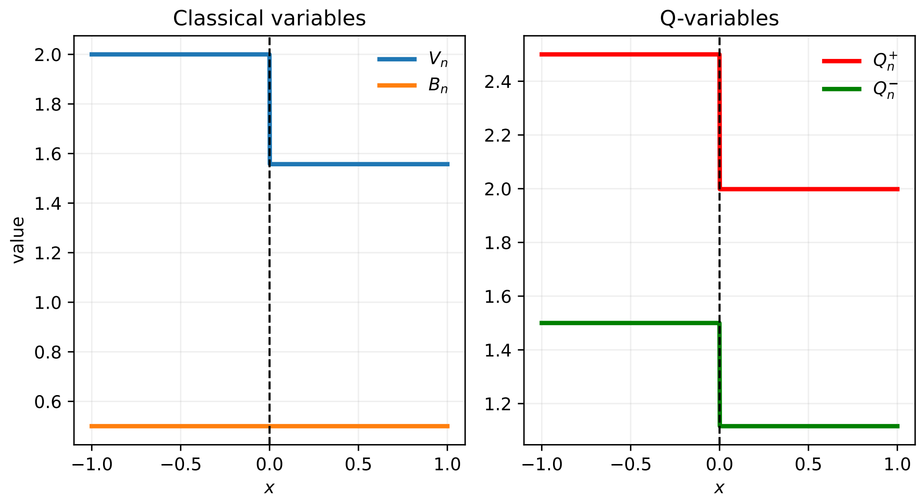

Figure 1: Riemann-type illustration of the same discontinuity in classical variables and in Q-variables. Classical variables show what jumps; Q-variables show the response of the selected wave branch to the jump.

Figure 1 illustrates the interpretation of the Q-variable jump conditions for a simple one-dimensional discontinuity. In classical variables, the discontinuity is described by the behaviour of V_n and B_n: the normal magnetic field is continuous, while the normal velocity changes across the interface. In Q-variables, the same physical jump appears through changes in Q_n^+ and Q_n^-. Classical variables therefore show which plasma quantities jump, while Q-variables show how the selected wave branch responds to the discontinuity.

The wave-branch parameter α also provides a link between wave diagnostics and discontinuity structure. For an Alfvénic branch,

$$\alpha^2 = \frac{1}{\mu\rho}, \qquad \frac{\alpha_1}{\alpha_2} = \sqrt{\frac{\rho_2}{\rho_1}}. \tag{18}$$

For a general propagating disturbance,

$$\alpha = \frac{\omega - \boldsymbol{k}\cdot\boldsymbol{V}_0}{\boldsymbol{k}\cdot\boldsymbol{B}_0}. \tag{19}$$

Thus, measuring the relevant wave branch on both sides of a discontinuity can constrain α_1 and α_2, which are linked to the jump in the background plasma state. This provides a route toward wave-based diagnostics of MHD discontinuities.

In conclusion, we have derived the Rankine-Hugoniot conditions directly in the Q-variable formalism by applying shock-limit identities to the Q-MHD system. The resulting jump relations preserve the full conservation-law structure of ideal MHD and recover the classical Rankine-Hugoniot conditions exactly. Their advantage is interpretive: the same discontinuity can be viewed through wave-aligned variables, making the response of selected MHD wave branches explicit. This formulation is therefore useful for analysing wave-shock interactions and for future applications in global MHD models where wave transport and discontinuity dynamics are coupled.

References

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