Particle acceleration at oblique shocks: the Vlasov–Fokker–Planck approach

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High energy particles carry a significant portion of the overall energy in many astrophysical systems and play a vital role in their dynamics. Shock fronts launched by supernovae, for example, convert a fraction of their kinetic energy into charged particles via interactions with magnetohydrodynamic (MHD) turbulence up- and downstream of the front itself. In addition, the compression of the magnetic field across an oblique shock contributes further to the acceleration. The Vlasov–Fokker–Planck approach models these processes by means of a partial differential equation (PDE) for the single-particle phase-space density of accelerated particles propagating in a prescribed background plasma flow. It contains terms that describe elastic scattering on turbulent fluctuations anchored in the background plasma as well as gyration about the background magnetic field and can be solved either by Monte-Carlo simulation of individual trajectories or by numerical discretisation of the PDE. The latter method generally requires the background flow to be continuous in space and time, which involves smoothing the sharp transition at a shock front. On this poster we present a new method that enables one to solve the PDE in the simple case of a stationary, oblique shock without smoothing the transition. Following [1], [2] and [3], we expand the angular dependence of the phase-space density in spherical harmonics, but instead of discretising the spatial variable, we diagonalise the system of equations and solve the resulting generalised eigenvalue problem. Comparison with the classical approach as implemented in `Sapphire++` and with Monte-Carlo simulations show good agreement for non-relativistic shocks and moderate values of the magnetisation parameter η' (ratio of the gyro-frequency to the scattering rate).

Physical setup

Shock wave

- Shock fronts are modelled as one-dimensional plasma flows characterised by their mean magnetic field $\mathbf{B}$ and their velocity $\mathbf{U} = U_s \mathbf{e}_x$.
- We focus on non-relativistic, $U_s \ll c$, and oblique, $\mathbf{B}^+ \cdot \mathbf{n} = \cos\psi^+ \in [0, 1]$, shock waves.
- In the eigenfunction method the shock wave is a true discontinuity.
- We compare the computed distribution functions with the results of `Sapphire++` [1], a PDE-based code that smooths the transition between the up- and downstream region, see Fig. 1.

Energetic particles

- Test particle limit: Particles do not change the state of the plasma, i.e. $\mathbf{U}$ and $\mathbf{B}$ are externally prescribed functions.
- Particles are isotropically and elastically scattered in the rest frame of the up- and downstream plasma.
- The scattering frequency is proportional to the inverse of the particle energy, i.e. $\nu \propto p^{-1}$.

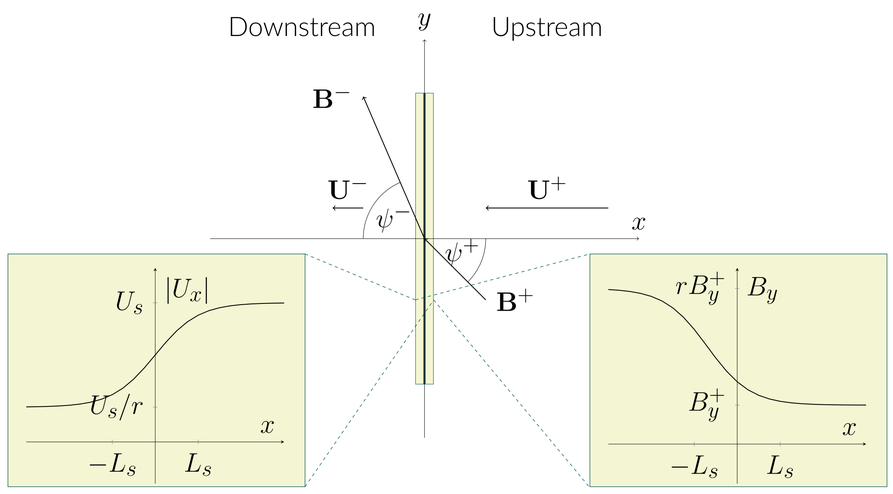

Figure 1. In `Sapphire++` a smooth velocity and $\mathbf{B}$-field profile are modelling the shock transition. In the eigenfunction method presented here, the shock is treated as a sharp discontinuity.

2D eigenfunction method

- We solve the Vlasov–Fokker–Planck (VFP) equation

$$\underbrace{\frac{\Gamma}{\nu'}\left(V'\hat{A}'_x - U_x\right)\frac{\partial f}{\partial x}}_{\text{spatial advection}} \underbrace{- \mathrm{i}\eta'(\cos\psi \hat{L}'_x + \sin\psi \hat{L}'_y) f}_{\text{gryo motion}} = \underbrace{\frac{1}{2}\Delta_{\theta',\varphi'} f}_{\text{isotropic scattering}} .$$

- $\eta' = \omega'_g/\nu' = \text{const.}$ is ratio of the gyro frequency to the scattering frequency.
- We solve the VFP equation in the up- and downstream independently and get an up- and downstream distribution function, f^+ and f^- respectively.
- We use the following ansatz for the distribution function,i.e.

$$f(x, p, \theta, \varphi) = N p^{-s} \sum_{i=0}^{N} f_i(x) Y_i(\theta, \varphi) .$$

- The expansion in spherical harmonics leads to a system of ordinary differential equations (ODE)s, which can be diagonalised solving the (generalised) eigenvalue problem

$$\left(\mathbf{A}_x - \frac{U_x}{V'}\mathbf{1}\right)^{-1} \left(\eta' \left[\cos\psi \boldsymbol{\Omega}_x + \sin\psi \boldsymbol{\Omega}_y\right] - \mathbf{C}\right) \mathbf{X} = \mathbf{X}\boldsymbol{\Lambda} .$$

- Using the eigenvectors $\mathbf{x}_j$ and eigenvalues Λ_j allows us to write the distribution function as

$$f = N p^{-s} \sum_j c_j \exp\left(\Lambda_j x\right) Q_j(\theta, \varphi) \quad \text{with} \quad Q_j \coloneqq \sum_i X_{ij} Y_i$$

- Because particles are not deflected by crossing the discontinuity, we can use Liouville's theorem to match the up- and downstream distribution at the shock, i.e.

$$f^+(x = 0, p^+, \theta^+, \varphi^+) = f^-(x = 0, p^-, \theta^-, \varphi^-) .$$

- This equality together with the Q-representation of f allows us to compute the spectral index s and the coefficients c_j of the scale-free solution when η' is not a function of momentum.

`Sapphire++` is a free and open-source code that also solves the VFP equation. However, the equation includes an additional fictitious force term, that changes the energy of the particles in agreement with the velocity profile when they cross the shock.

- It also uses a spherical harmonic expansion of f.
- It discretises the resulting system of PDEs in x and p using the discontinuous Galerkin method.
- In particular, this implies that it does not make any assumptions about the p-dependence of f.

Results

Spectral index

- The eigenfunction method was used to compute the spectral index s for varying shock obliquity $\cos\psi^+$, shock velocity U^+ and scattering regimes η.

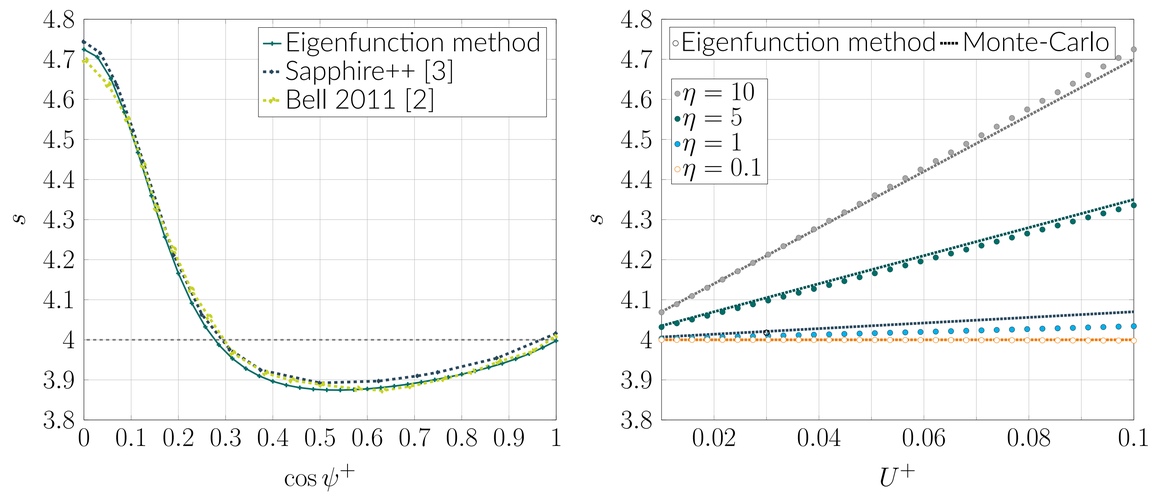

Figure 2. *Left:* Spectral index as a function of shock obliquity for fixed $U^+ = 0.1$ and $\eta = 10$. The eigenfunction method is compared to two PDE-based methods that use continuous velocity and magnetic field profiles.
Right: Spectral index as a function of shock velocity and scattering regimes for a perpendicular shock ($\cos\psi^+ = 0$). The eigenfunction method (dots) is compared to a Monte-Carlo method [4] (dotted lines).

- The good agreement between the two "classical" approaches and the eigenfunction method indicates that the smoothing of the shock transition does not qualitatively change the energy distribution of the particles.
- The systematic offset towards slightly smaller spectral indices may be explained with the influence of the shock width.

Anisotropy at the shock front

- The spectral index s depends on the anisotropy of the distribution function at the shock. We compute the spatial dependence of the (higher order) expansion coefficients at a fixed p.

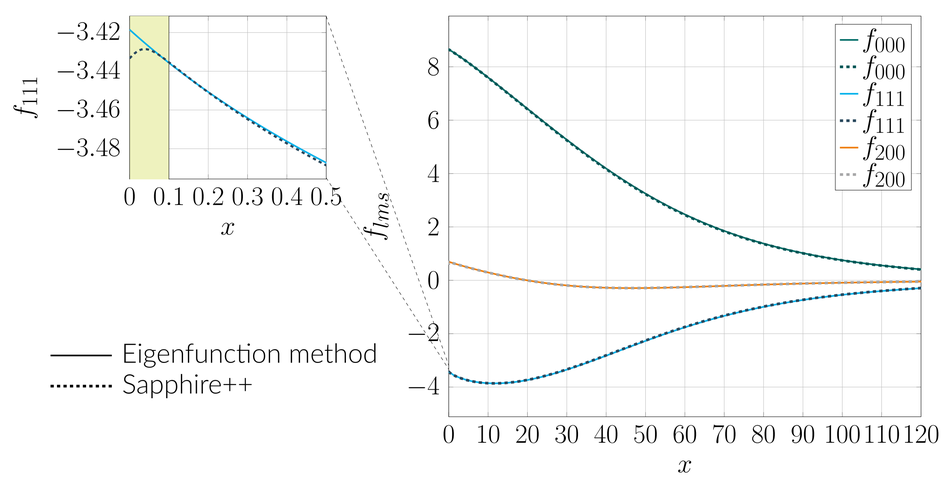

Figure 3. The spatial dependence of upstream expansion coefficients for a fixed p and a perpendicular shock, $U^+ = 0.1$ and $\eta = 5$. Zoom into the shock transition region. The eigenfunction method (solid lines) is compared to `Sapphire++` (dashed lines).

- The expansion coefficients agree outside the shock transition. The anisotropy is the same for continuous and discontinuous velocity/magnetic field profiles.

Discussion

- We tested the eigenfunction method up to values of $\eta \leq 10$: works well in general, though it shows defects for certain combinations of parameters.
- Avoiding the spatial discretisation of the PDE makes the eigenfunction method computationally fast, which enables parameter scans and rapid modelling.
- Using basis functions other than spherical harmonics could enable it to treat also relativistic shocks.

References

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