Particle acceleration at oblique shocks: the Vlasov–Fokker–Planck approach

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Introduction

The Vlasov–Fokker–Planck (VFP) approach may be used to model the acceleration processes of high energy particles at oblique shocks in a prescribed background plasma flow by means of a partial differential equation (PDE) for their single-particle phase-space density. This can be solved either by Monte-Carlo simulation of individual trajectories or by numerical discretisation. The latter method generally requires the background flow to be continuous in space and time, which involves smoothing the sharp transition at a shock front. Here we present a new method that enables one to solve the PDE in the simple case of a stationary, oblique shock without smoothing the shock transition. The method uses 2D-eigenfunctions to expand the angular dependence of the particles' phase-space density, as investigated in the context of perpendicular shocks [1]. The method follows [2] and [3], and uses a spherical harmonic expansion of the phase-space density, though it does not discretise the spatial variable. The expansion transforms the PDE into a system of ordinary differential equations (ODE) whose general solution can be obtained via diagonalisation. The eigenvectors of the corresponding eigenvalue problem are spherical harmonic representations of the 2D-eigenfunctions.

Physical setup

We aim to compute the phase-space density function f of test particles that are accelerated at astrophysical shocks. In the shock rest frame, the ambient plasma moves along the x-axis with constant velocity $-U^+$ for $x > 0$ (upstream) and $-U^- = -U^+/4$ for $x < 0$ (downstream). In the local rest frame of the plasma, the electric field vanishes and the ambient magnetic field is taken to be uniform both upstream and downstream. It lies in the $x-y$ plane, making an angle ψ with respect to the shock normal in the upstream. The Rankine–Hugoniot conditions for a weakly-magnetised shock are used to determine the magnetic field downstream.

2D eigenfunction method

The single-particle distribution function f is modelled with the Vlasov–Fokker–Planck (VFP) equation in mixed coordinates, i.e.

$$\Gamma\left(V'\hat{A}'_x - U\right)\frac{\partial f}{\partial x} - \mathrm{i}\omega'_g(\cos\psi\hat{L}'_x + \sin\psi\hat{L}'_y) = \frac{\nu'}{2}\Delta_{\theta',\varphi'} f\,, \tag{1}$$

where primed quantities are defined in the rest frame of the background plasma and x in the shock rest frame. We call $A'_x = \cos\theta'$ a direction operator and $\hat{L}'_x$ and $\hat{L}'_y$ are the components of the angular momentum operator $\mathbf{L}' = -\mathrm{i}\mathbf{p}' \times \partial/\partial\mathbf{p}'$, for details see [4]. We assume Bohm scaling of the scattering frequency, namely $\nu \propto p^{-1}$. This implies that the magnetisation parameter $\eta' = \omega'_g/\nu'$ is a constant. Henceforth, we drop the primes for ease of reading.

Eq. (1) is solved independently on each side of the shock front with $U = U^{\pm}$, to find the up- and downstream distribution functions f^+ and f^- respectively. To do this we expand f using (real) spherical harmonics and assume a power-law momentum dependence. The latter assumption is justified, because eq. (1) is momentum-scale free. This turns the PDE into a system of ODEs, i.e.

$$\left(\mathbf{A}_x - \frac{U}{V}\mathbf{1}\right)\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x^*} - \eta\left(\cos\psi\mathbf{\Omega}_x + \sin\psi\mathbf{\Omega}_y\right)\mathbf{f} = -\mathbf{C}\mathbf{f}, \tag{2}$$

where $\mathbf{f} = (f_0, f_1, \dots)$ is a vector whose elements are the expansion coefficients of the spherical harmonic expansion and $\mathbf{A}_x$, $\mathbf{\Omega}_x, \mathbf{\Omega}_y$ and $\mathbf{C}$ are matrix representations of the operators appearing in eq. (1), see [4]. We also introduced the dimensionless variable $x^* = x\nu/\Gamma V$. This system of ODEs can be diagonalised by solving the (generalised) eigenvalue problem

$$\left(\mathbf{A}_x - u_x\mathbf{1}\right)^{-1}\left(\eta\left[\cos\psi\mathbf{\Omega}_x + \sin\psi\mathbf{\Omega}_y\right] - \mathbf{C}\right)\mathbf{X} = \mathbf{X}\mathbf{\Lambda}. \tag{3}$$

Hence, the *general* solution of eq. (2) is $f_i = X_{ij}c_j\exp\left(\Lambda_j x^*\right)$, where we implicitly sum over j, allowing us to write the distribution function f as

$$f = Np^{-s}f_i(x^*)Y_i = Np^{-s}X_{ij}c_j\exp\left(\Lambda_j x^*\right)Y_i = Np^{-s}c_j\exp\left(\Lambda_j x^*\right)Q_j, \tag{4}$$

with $Q_j \equiv X_{ij}Y_i$. The Q's are the 2D-eigenfunctions explored in [1], generalised to oblique shocks and represented in terms of the spherical harmonics Y_i. Note that solving the eigenvalue problem (3) using the conjugate transpose of the appearing matrices yields a set of functions that is orthogonal to the Q_j functions with respect to the weight function $w = \cos\theta - U/V$. We denote them with $\bar{Q}_j$ and call them *adjoint* Qs.

We exploit the representation of f given in eq. (4) to derive a *particular* solution, i. e. to determine the unknown coefficients c_j and the spectral index s. Physics helps us in this endeavour: firstly, the number of energetic particles must remain finite as $x \to \pm\infty$ and, secondly, the trajectories of the particles are undisturbed by the shock front. The first fact implies that we have to set those upstream coefficients $c_j^+ = 0$ that correspond to eigenvalues whose real part is positive. This is also true for the downstream coefficients, though with reversed sign. The second fact allows us to derive a linear system of equations that determines the remaining coefficients c_j^+, since it implies that $f^+(x=0,\mathbf{p}^+) = f^-(x=0,\mathbf{p}^-)$, where $\mathbf{p}^+$ and $\mathbf{p}^-$ denote the same physical momentum related through a Lorentz transformation. This leads to

$$\int \bar{Q}_i^-(\theta^-,\varphi^-)\left(\frac{p^+}{p^-}\right)^{-s}\left(\cos\theta^- - \frac{U^-}{V^-}\right) Q_j^+(\theta^+,\varphi^+)\mathrm{d}\Omega^- c_j^+ = d_0\delta_{i0}\,, \tag{5}$$

where both i and j correspond to eigenvalues that are positive or zero, since the downstream distribution function may not contain Q's corresponding to eigenvalues with negative real part. We label the isotropic eigenfunction with $i=0$, so that $\bar{Q}_0^-$ corresponds to the isotropic part of $f^-(x=0,\mathbf{p}^-)$, which is normalised to d_0 and determines the distribution far downstream, since $f^-(x,\mathbf{p}^-) \to d_0$ as $x \to -\infty$. The linear system of equations (5) for the coefficients depends non-linearly on the spectral index s. If we require that no particles are incoming from far upstream, then c_0^+ has to equal zero. This allows us to use a root finder to determine s and the non-zero expansion coefficients c_j^+, cf. [5].

Results

We computed the spectral index s for varying shock obliquity $\cos\psi^+$, shock velocity U^+ and scattering regimes η and compared the results with `Sapphire++` and, for cross-validation, with a Monte-Carlo code that investigated particle acceleration at a perpendicular shock [1].

In the left panel of Fig. 1 we show that the computed spectral indices as a function of shock obliquity are in good agreement with `Sapphire++` [6] and with [2]. Both of these codes use a continuous shock profile, but the latter prescribes a power-law energy dependence, in contrast to `Sapphire++` where the energy dependence is computed. The good agreement indicates that the smoothing of the shock transition does not qualitatively change the energy distribution of the particles. However, a systematic offset towards slightly smaller spectral indices can be observed and may be explained as the effect of the finite shock width.

In the right panel of Fig. 1 the spectral indices of the eigenfunction method (dots) follow closely the results of the Monte-Carlo simulations (dotted lines) for all tested magnetisation parameters and varying shock speeds. We take this as evidence that the results are method independent.

The spectral index s depends on the anisotropy of the distribution function at the shock. In order to check how it is influenced by the continuous velocity and magnetic field profile, we compute the spatial dependence of the expansion coefficients.

The result is shown in Fig. 2: The expansion coefficients agree outside the shock transition, see the zoom into the shock transition region on the left. The anisotropy is the same for continuous and discontinuous velocity/magnetic field profiles.

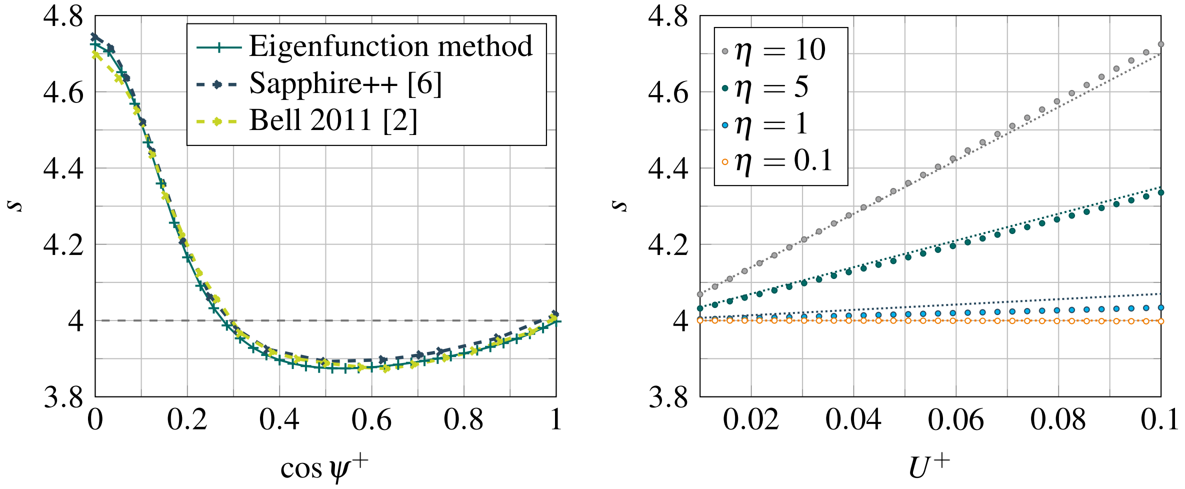

Figure 1: *Left:* Spectral index as a function of shock obliquity for fixed $U^+ = 0.1$ and $\eta = 10$. *Right:* Spectral index as a function of shock velocity and scattering regimes for $\cos\psi^+ = 0$.

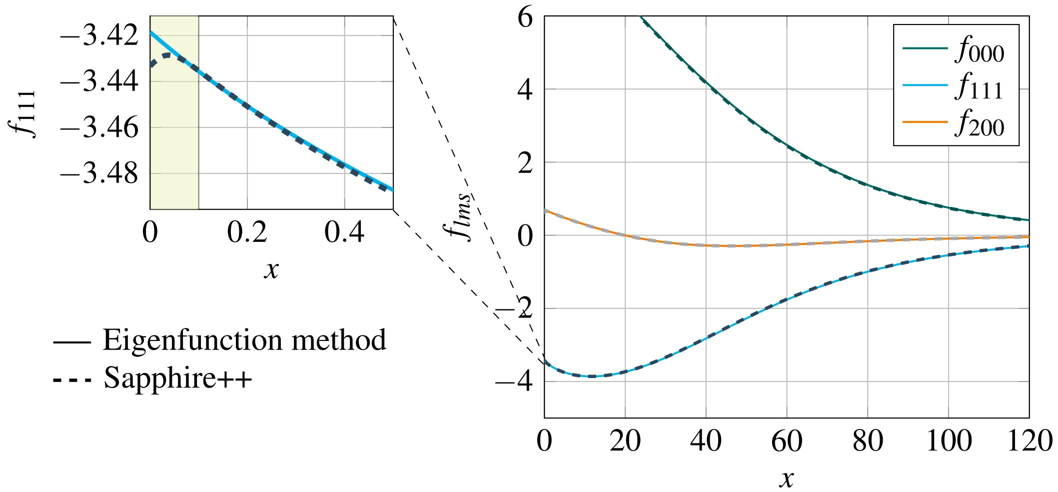

Figure 2: The spatial dependence of upstream expansion coefficients for a fixed p, $\cos\psi^+ = 0$, $U^+ = 0.1$ and $\eta = 5$. Zoom into the shock transition region: the shaded area marks the shock width.

Discussion

We tested the eigenfunction method up to values of $\eta \leq 10$: it works well in general, though it shows defects for certain combinations of parameters. We remark that avoiding the spatial discretisation of the PDE makes the eigenfunction method computationally fast, which enables parameter scans and rapid modelling. The method can also be implemented using basis functions other than spherical harmonics, which should enable it to treat also relativistic shocks [7].

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