Fluctuation Theorem for Collisionless Processes in Kinetic Plasma Systems

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Introduction

Collisionless plasmas exhibit a wide variety of collective phenomena through long-range electromagnetic interactions. Among them, Landau damping [1] has been extensively studied as a fundamental mechanism for stabilizing instabilities, saturating fluctuations, and heating particles in space and fusion plasmas. Although Landau damping is governed by the time-reversal-symmetric Vlasov equation, it appears macroscopically irreversible. This apparent irreversibility arises because phase mixing transfers information from low-order velocity moments to increasingly fine velocity-space structures, while the Gibbs entropy defined from the velocity distribution function remains conserved. In Ref. [2], the linearized Vlasov-Poisson system of equations is transformed into Schrödinger form, enabling the fluctuation theorem to be applied to Landau damping. The fluctuation theorem [3, 4], derived from reversible microscopic dynamics, relates the probabilities of entropy production and entropy reduction, and thereby provides a microscopic basis for the second law of thermodynamics. In this study, the fluctuation is formulated for classical systems governed by Schrödinger equations with time-reversal symmetry [5]. As examples of such systems, the linear Vlasov-Poisson system and the linear gyrokinetic system are investigated.

Schrödinger Equation and Fluctuation Theorem

We consider a classical system whose state at time τ is represented by the ket vector $|\psi(\tau)\rangle$, which satisfies a time-reversal-symmetric Schrödinger equation ($\hbar = 1$),

$$i\frac{d}{d\tau}|\psi(\tau)\rangle = \widehat{H}|\psi(\tau)\rangle, \tag{1}$$

where $\widehat{H}$ is a Hermitian Hamiltonian operator. Here, $|\psi(\tau)\rangle$ represents a classical state, not a quantum-mechanical probability amplitude associated with measurement. We consider the vector space spanned by the N_c ket vectors $\{|n\rangle\}_{n=0,1,\cdots,N_c-1}$ and represent the state vector $|\psi(\tau)\rangle$ in this space as the N_c-dimensional complex vector, $\psi(\tau) \equiv {}^t[\psi_0(\tau), \psi_1(\tau), \cdots, \psi_{N_c-1}(\tau)]$, where $\psi_n(\tau) = \langle n|\psi(\tau)\rangle$ for $n = 0, 1, \cdots, N_c - 1$. Here, we impose the orthonormality and closure conditions, $\langle n|n'\rangle = \delta_{nn'}$ and $\sum_{n=0}^{N_c-1} |n\rangle\langle n| = \widehat{1}$. The Schrödinger equation is expressed for $\psi(\tau)$ as

$$i\frac{d}{d\tau}\psi(\tau) = \mathbf{H}\psi(\tau), \tag{2}$$

where $\mathbf{H}$ is a Hermitian $N_c \times N_c$ Hamiltonian matrix defined from the Hamiltonian operator $\widehat{H}$ by $\mathbf{H} \equiv [H_{nn'}]_{n,n'=0,1,\cdots,N_c-1}$ and $H_{nn'} \equiv \langle n|\widehat{H}|n'\rangle$.

We now treat the initial vector $\psi(0)$ as a random variable. Then, the vector $\psi(\tau)$ at time τ, which is uniquely determined by $\psi(0)$, is also a random variable. The probability that the real and imaginary parts of $\psi_n(\tau) \equiv \psi_{r,n}(\tau) + i\psi_{i,n}(\tau)$ $(n = 0, 1, 2, \cdots, N_c - 1)$ lie within the infinitesimal intervals $[\psi_{r,n}, \psi_{r,n} + d\psi_{r,n})$ and $[\psi_{i,n}, \psi_{i,n} + d\psi_{i,n})$, respectively, is given by $P(\psi; \tau)\, d\Gamma$, where the volume element is defined as $d\Gamma \equiv \prod_{n=0}^{N_c-1} d\psi_{r,n} d\psi_{i,n}$. Then, $P[\psi(\tau); \tau] = P[\psi(0); 0]$ holds, which is the analogue of Liouville's theorem in Hamiltonian mechanics. The stochastic relative entropy of the distribution $P[\psi(\tau); \tau] = P[\psi(0); 0]$ with respect to $P[\psi(\tau); 0]$, is defined by

$$\Delta S[\psi(0); \tau] \equiv \log\left[\frac{P[\psi(\tau); \tau]}{P[\psi(\tau); 0]}\right] \equiv \log\left[\frac{P[\psi(0); 0]}{P[\psi(\tau); 0]}\right], \tag{3}$$

where $\psi(\tau)$ is related to $\psi(0)$ by $\psi(\tau) = \exp(-i\tau\mathbf{H})\psi(0)$. The probability density function $P(\Delta S)$ is defined such that $P(\Delta S)d(\Delta S)$ gives the probability that the stochastic relative entropy $\Delta S[\Psi(0); \tau]$ lies in the infinitesimal interval $[\Delta S, \Delta S + d(\Delta S))$. The fluctuation theorem states that, under the condition that the initial distribution $P[\psi(\tau); 0]$ is symmetric with respect to time-reversal operation,

$$\frac{P(\Delta S)}{P(-\Delta S)} = \exp(\Delta S). \tag{4}$$

A detailed proof of the theorem is given in Ref. [5].

Linear Vlasov-Poisson System

We consider the linear Vlasov-Poisson system, in which a one-dimensional electrostatic perturbation of a Maxwellian plasma is investigated [2]. The solution $f_1(x, v, t)$ of the linearized Vlasov-Poisson system of equations is written as $f_1(x, v, t) = \mathrm{Re}[f_1(k, v, t)\exp(ikx)]$, and the normalized time and velocity are defined by $\tau \equiv kv_T t$ and $\xi \equiv v/v_T$, respectively, where $v_T \equiv \sqrt{2T/m}$. We use the Hermite polynomials $H_n(\xi)$ $(n = 0, 1, 2, \cdots)$ to expand $f_1(k, v, t)$ as

$$f_1(k, v, t) = (n_0/v_T)\pi^{-1/2} e^{-\xi^2} \sum_{n=0}^{\infty} (1 + \kappa^{-2}\delta_{n0})^{-1/2} \psi_n(\tau) H_n(\xi)/(2^n n!). \tag{5}$$

Using $\psi_n(\tau)$ shown above, we define the ket state vector by $|\psi(\tau)\rangle = \sum_{n=0}^{\infty} \psi_n(\tau)|n\rangle$ where $\{|n\rangle\}_{n=0,1,2,\cdots}$ give basis vectors satisfying $\langle n|n'\rangle = \delta_{nn'}$, $\sum_{n=0}^{\infty} |n\rangle\langle n| = \widehat{1}$, $\langle \xi|n\rangle = h_n(\xi) \equiv \pi^{-1/4} e^{-\xi^2/2} H_n(\xi)/(2^n n!)^{1/2}$, and $\psi_n(\tau) = \langle n|\psi(\tau)\rangle$. Then, it is important to note that the invariant $D[f_1]$ of this system is proportional to the squared norm,

$$D[f_1] \equiv \int_{-L/2}^{+L/2} \frac{dx}{L} \left[\frac{[E(x,t)]^2}{8\pi n_0 T} + \frac{1}{n_0}\int_{-\infty}^{+\infty} dv\, \frac{[f_1(x, v, t)]^2}{2f_0(v)}\right] = \frac{1}{4}\langle \psi(\tau)|\psi(\tau)\rangle = \frac{1}{4}\sum_{n=0}^{\infty} |\psi_n(\tau)|^2, \tag{6}$$

from which it follows that $\psi(\tau)$ should satisfy the Schrödinger equation as in Eq. (1). In the present case, the Hamiltonian operator is given by

$$\widehat{H} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \sqrt{n + 1 + \kappa^{-2}\delta_{n0}} \Big(|n+1\rangle\langle n| + |n\rangle\langle n+1|\Big). \tag{7}$$

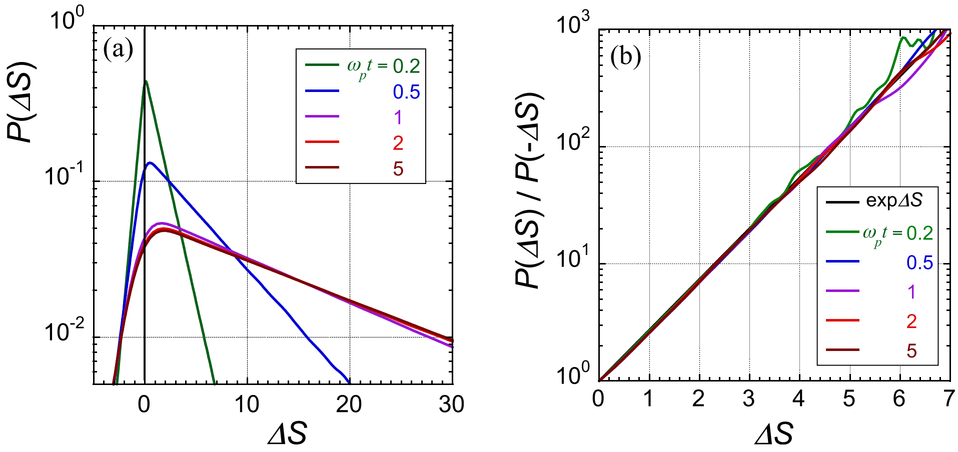

Figure 1: (a) Probability density function $P(\Delta S)$ of the stochastic relative entropy and (b) the ratio $P(\Delta S)/P(-\Delta S)$ obtained numerically at time $\omega_p t = 0.2$, 0.5, 1, 2, and 5. Reproduced from Ref. [2].

The Hamiltonian eigenvectors $|\mathrm{CVK},\zeta\rangle$ $(-\infty < \zeta < +\infty)$ corresponding to Case–Van Kampen modes [6] can be derived [2, 5], and they satisfy $\widehat{H}|\mathrm{CVK},\zeta\rangle = \zeta|\mathrm{CVK},\zeta\rangle$, $\langle\mathrm{CVK},\zeta|\mathrm{CVK},\zeta'\rangle = \delta(\zeta-\zeta')$, and $\int_{-\infty}^{+\infty}\mathrm{CVK},\zeta\rangle d\zeta\langle\mathrm{CVK},\zeta| = \widehat{1}$. We now use a discrete set of eigenvectors $|\mathrm{CVK},\zeta_j\rangle$ $(j=0,1,\cdots,N_{\mathrm{cvk}}-1)$ with N_{cvk} real eigenvalues ζ_j defined by the condition $\langle N_{\mathrm{cvk}}|\mathrm{CVK},\zeta_j\rangle = 0$ that gives the N_{cvk}th-order algebraic equation for ζ_j. Then, we consider the solution given by $|\psi(\tau)\rangle = \sum_{j=0}^{N_{\mathrm{cvk}}-1} c_j(\tau)|\psi_{\mathrm{CVK},\zeta_j}\rangle$, where $c_j(\tau) = c_j(0)e^{-i\zeta_j\tau}$. This representation is particularly useful for formulating the fluctuation theorem because the time evolution is explicitly unitary in the truncated state-vector space. Extracting the first N_{cvk} components, $\{\psi_n(\tau)\}_{n=0,1,\cdots,N_{\mathrm{cvk}}-1}$, from the solution given above, we define the N_{cvk}-dimensional complex column vector, $\psi(\tau) \equiv {}^t[\psi_0(\tau),\psi_1(\tau),\cdots,\psi_{N_{\mathrm{cvk}}-1}(\tau)]$, which is found to satisfy $i\,d\psi(\tau)/d\tau = \mathbf{H}\psi(\tau)$, where $\mathbf{H} = [H_{nn'}]$ is the $N_{\mathrm{cvk}}\times N_{\mathrm{cvk}}$ Hamiltonian matrix whose elements are defined by $H_{nn'} \equiv \langle n|\widehat{H}|n'\rangle$.

We assume that the initial state vector $\psi(0)$ is randomly distributed according to the probability density function given by $P[\psi(0);0] = Z^{-1}\exp[-\sum_{n=0}^{N_{\mathrm{cvk}}-1}\beta_n|\psi_n(0)|^2]$. Here, $Z \equiv \int d\Gamma(0)\exp[-\sum_{n=0}^{N_{\mathrm{cvk}}-1}\beta_n|\psi_n(0)|^2]$, and $\beta_n = \beta_0/\rho$ for $n = 1,2,\cdots,N_{\mathrm{cvk}}-1$, where $\beta_0 > 0$ and $0 < \rho < 1$. It is shown in this case that the stochastic relative entropy measures the transfer of fluctuation intensity from the electric-field component, represented by the lowest Hermite state ψ_0, to higher-order Hermite states ψ_n $(n \geq 1)$. Figure 1 shows the probability density function $P(\Delta S)$ of the stochastic relative entropy and the ratio $P(\Delta S)/P(-\Delta S)$ obtained numerically for $\rho = 1/20$ at time $\omega_p t = 0.2$, 0.5, 1, 2, and 5. The fluctuation theorem, Eq. (4), is confirmed with good accuracy in Fig. 1(b).

Linear Gyrokinetic System

The governing equations for a linear gyrokinetic system in a uniform background magnetic field are cast in Schrödinger form with time-reversal symmetry, using an invariant analogous to that of the linear Vlasov-Poisson system to define the squared norm of the state vector [5].

Therefore, the fluctuation theorem also applies to this system to this system. The state-vector space is written as the tensor product of species, perpendicular-velocity, and parallel-velocity spaces. The state vector decomposes into two mutually orthogonal components: one associated with self-consistently produced electromagnetic fluctuations, and the other corresponding to rapidly decaying ballistic modes.

Conclusions

This paper presents a framework in which fluctuation theorems can be applied to collisionless kinetic plasma systems by reformulating their linear dynamics in Schrödinger form. In this formulation, the fluctuation theorem holds for the stochastic relative entropy defined from the probability density functional of the particle velocity distribution function. We first consider the linear Vlasov-Poisson system, defining the corresponding state vector such that its squared norm is proportional to a time-independent quadratic invariant. The stochastic relative entropy is interpreted as entropy generation associated with Landau damping, in which electric-field energy is transferred from the lowest Hermite state to higher-order Hermite states acting as thermal reservoirs. The second example is a linear electromagnetic gyrokinetic system in a uniform background magnetic field. Its governing equations can be written in Schrödinger form with time reversal symmetry, so the fluctuation theorem also holds. The present results provide a nonequilibrium statistical-mechanical perspective on collisionless plasma dynamics and offer useful examples for future quantum-computing applications to plasma simulations.

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