A Normalized Hessian-Eigenvalue Measure of Confinement in Single- and Two-Species Space-Charge Equilibria

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The structure that governs confinement in a space-charge potential is characterized here by the local second derivatives of the potential rather than by a global well depth. These second derivatives form the Hessian of the potential, whose trace is fixed by the local net charge density through Poisson's equation. The Hessian-based method for characterizing space-charge based plasma confinement is intended to be applicable and most useful for complex geometries. For initial studies, the method is applied to geometries with symmetries that reduce equilibrium solutions to one dimension. An eigenvalue of the Hessian is evaluated at a critical point of the potential for self-consistent single- and two-species non-neutral plasma equilibria. In the single-species equilibrium, as the plasma size increases to values much larger than the smallest Debye length, the critical-point eigenvalue approaches zero and a normalized expression for the electric field at the plasma's edge approaches a geometry-independent value. In the two-species equilibrium, a confined second species reduces the critical-point eigenvalue by several orders of magnitude compared to the single-species equilibrium.

Introduction

The confinement and equilibrium properties of non-neutral plasmas have been studied extensively [1, 2]. For the equilibria considered here, global characterizations such as the normalized well depth or statistical results from particle trajectory simulations require information from across the entire potential profile [3]. Here we introduce a complementary measure built from the eigenvalues of the potential's Hessian, the local second derivatives along the principal directions. We define a single dimensionless measure from these eigenvalues and evaluate it for two established non-neutral plasma equilibria, a single-species plasma [3, 4] and a two-species electron–positron plasma [3].

Normalized Hessian Eigenvalue

The normalized electrostatic potential $\psi(\mathbf{x})$ at position $\mathbf{x}$ of a space-charge distribution is a solution of Poisson's equation [5]. The treatment is similar for a single-species plasma and for the species that provides space-charge based confinement in a two-species plasma. The potential is normalized such that ψ is equal to a particle's potential energy divided by plasma

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temperature in energy units. Also, lengths are normalized by the Debye length at the plasma edge, so Poisson's equation takes the form $\nabla^2\psi = -\rho$, where ∇^2 is the Laplacian and $\rho = \rho_{\text{phys}}/(q_0 n_0)$ is the normalized charge density, with ρ_{phys} the physical charge density, q_0 the signed charge of that species, and n_0 its edge number density. The local second derivatives of the potential form the Hessian $H_{ij} = \partial_i\partial_j\psi$, where ∂_i is the partial derivative with respect to the ith Cartesian coordinate and the indices i and j run over the spatial dimensions. This real symmetric tensor has eigenvalues λ_k ($k = 1,2,3$), the second derivatives of ψ along the principal directions. The trace of a real symmetric tensor equals the sum of its eigenvalues regardless of the coordinate axes, so this sum equals the Laplacian $\nabla^2\psi$ and is fixed by the net charge density through Poisson's equation,

$$\operatorname{Tr}(H) = \lambda_1 + \lambda_2 + \lambda_3 = \nabla^2\psi = -\rho(\mathbf{x}). \tag{1}$$

Equation (1) holds whether or not the off-diagonal elements H_{ij} ($i \neq j$) vanish. At locations where $\rho = 0$, the eigenvalues sum to zero and therefore cannot all be strictly positive or all strictly negative, consistent with Earnshaw's theorem [6].

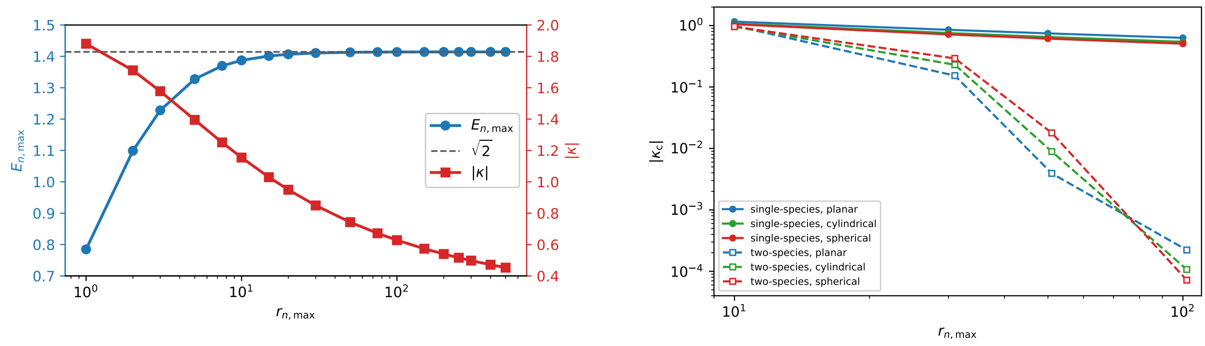
A location is referred to here as a critical point if the electric field there is zero, $\nabla\psi = 0$. The normalized Hessian eigenvalue at a critical point $\mathbf{x}_0$ is defined as

$$\kappa = \frac{\lambda\, r_{n,\max}^2}{\Delta\psi}, \tag{2}$$

where λ is a Hessian eigenvalue at the critical point, $r_{n,\max}$ is a measure of the normalized system size, and $\Delta\psi$ is a measure of the space-charge based well depth. Because ψ and the coordinates are normalized, λ, $r_{n,\max}$, and $\Delta\psi$ are dimensionless, so κ is a dimensionless measure of the eigenvalue λ relative to the well depth divided by the square of the system size. For a particle of the species used to normalize ψ, a small displacement along a principal direction is restoring when $\lambda > 0$. For a particle of charge q, the condition is $(q/q_0)\lambda > 0$. The single-species plasma is considered here to be a positron plasma, and the two-species plasma is considered to be a positron plasma confined by the space charge of an electron plasma.

Equilibria and Numerical Evaluation

The single- and two-species equilibria considered here are the self-consistent solutions of the normalized Poisson–Boltzmann equations in planar, cylindrical, and spherical geometry ($\alpha = 1,2,3$) on $0 \le r_n \le r_{n,\max}$, where r_n is the normalized radial coordinate, with regularity at the origin and $\psi(r_{n,\max}) = 0$ [3, 4]. For an electron–positron plasma, let N_n denote the ratio of the positron density at the plasma's center to the electron density at the plasma's edge. Also, let T_n denote the electron-to-positron temperature ratio. The normalized Hessian eigenvalue



(a) (b)

Figure 1: (a) Normalized maximum electric field $E_{n,\max}$ (left axis) and central $|\kappa|$ (right axis) versus $r_{n,\max}$ for the planar single-species equilibrium. Over $r_{n,\max} = 1$ to 500, $E_{n,\max}$ rises toward the geometry-independent value $\sqrt{2}$ (dashed line) while $|\kappa|$ falls. Edge-field data from Ref. [4]. (b) Central $|\kappa|$ versus $r_{n,\max}$ over 10 to 100 for the single-species (solid) and two-species (dashed) equilibria in all three geometries. At the largest system sizes shown, the two-species values are three to four orders of magnitude smaller. The two-species curves use $N_n = 10^{-2}$ and $T_n = 5$.

is evaluated over $r_{n,\max} = 1$–500 for the single-species system and over $N_n = 10^{-3}$–0.3 and $T_n = 1$–20 at $r_{n,\max}$ up to 300 for the two-species system. At the central critical point of these symmetric equilibria the Hessian is diagonal, and the α eigenvalues associated with the active symmetry dimensions are equal while the remaining $3-\alpha$ eigenvalues are zero, so κ reduces to a single scalar for each geometry. This common eigenvalue λ is obtained from the regular origin limit of Eq. (1) as $\lambda = -\rho(0)/\alpha$, where $\rho(0)$ is the net charge density at the origin.

Single-Species System

A single positive species confined externally along its edge, with normalized number density $n(r_n) = e^{-\psi(r_n)}$ and $\psi = 0$ at $r_n = r_{n,\max}$ [3, 4], has one interior critical point at $r_n = 0$. This point is a potential maximum, and its isotropic Hessian eigenvalue follows from Eq. (1) and L'Hôpital's rule as $\lambda = -n(0)/\alpha$, where $n(0)$ is the central density and $\alpha = 1,2,3$ is the geometry index (planar, cylindrical, spherical). The normalized Hessian eigenvalue is

$$\kappa = -\frac{n(0)}{\alpha}\frac{r_{n,\max}^2}{\Delta\psi}, \tag{3}$$

and $|\kappa|$ falls monotonically with normalized system size, from $1.16, 1.07, 1.05$ (planar, cylindrical, spherical) at $r_{n,\max} = 10$ to $0.63, 0.54, 0.51$ at $r_{n,\max} = 100$. As the system grows, $\Delta\psi$ increases only logarithmically while $n(0)$ falls exponentially [4].

Over the same range the normalized maximum electric field $E_{n,\max}$ approaches the geometry-independent value $\sqrt{2}$ [4] while $|\kappa|$ falls toward zero (Fig. 1a). The field is a property of the plasma boundary, whereas κ is evaluated at the central critical point, so the two move in opposite directions as the system grows.

Two-Species System

A confined positron population held in the well of an electron space charge [3] satisfies

$$\frac{1}{r_n^{\alpha-1}}\frac{d}{dr_n}\left(r_n^{\alpha-1}\frac{d\psi}{dr_n}\right) = -e^{-\psi} + N_n\, e^{T_n(\psi-\psi_0)}, \qquad \alpha = 1,2,3, \tag{4}$$

where $\psi_0 = \psi(0)$ is the normalized potential at the geometric center of the system. The electron species provides the normalization, so $q_0 = -e$, the confining electron density is $n_e = e^{-\psi}$, and the confined positron density is $n_p = N_n\, e^{T_n(\psi-\psi_0)}$. Thus $\rho = n_e - n_p$ and Eq. (4) is $\nabla^2\psi = -\rho$, consistent with Eq. (1). In the limit of large normalized system size, the equilibrium forms a quasi-neutral core, and the well depth saturates at the value $\Delta\psi_0 = -\ln N_n$ [3].

As the quasi-neutral core develops, $|\lambda|$ becomes as small as 10^{-7}, and $|\kappa|$ falls several orders of magnitude below its single-species value, reaching approximately 10^{-3} or smaller for the larger systems in Fig. 1b. Because the core is quasi-neutral, the net charge and, by Eq. (1), the sum of the eigenvalues are concentrated in the electron-dominated boundary layer, leaving the potential nearly flat at the center. The central κ therefore characterizes this local second derivative, not the total energy required for escape.

Conclusion

A single normalized Hessian eigenvalue of the space-charge potential, $\kappa = \lambda\, r_{n,\max}^2/\Delta\psi$, characterizes the local second derivative of the potential at a critical point. In these symmetric equilibria, the trace identity $\mathrm{Tr}(H) = -\rho$ relates its value to the local net charge density. In the single-species equilibrium $|\kappa|$ falls toward zero as the normalized maximum electric field approaches the geometry-independent value $\sqrt{2}$. In the two-species equilibrium a quasi-neutral core reduces $|\kappa|$ by several orders of magnitude, while the Hessian trace is concentrated in the charged boundary layer. A redistribution of charge, through a larger system or a second species, changes the local net charge density and, with it, a local Hessian eigenvalue that global measures do not capture.

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